# LIMITS AND CONTINUITY FOR FUNCTIONS OF TWO OR MORE VARIABLES

LIMITS ARE MORE SUBTLE (AND MORE INTERESTING) FOR FUNCTIONS OF MORE THAN ONE VARIABLE.

RECALL : GIVEN $y = f(x)$ AND A REAL NUMBER $x_0$ WITH $f$ DEFINED AT LEAST "NEAR" $x_0$ ( $x_0$ ) WE SAY THAT

$$\lim_{x \to x_0} f(x) = L$$

IF $f(x)$ CAN BE MADE AS CLOSE AS WE LIKE TO $L$ BY CHOOSING $x$ SUFFICIENTLY CLOSE (BUT NOT EQUAL) TO $x_0$.

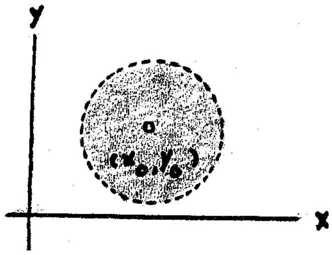
$\lim_{x \to x_0} f(x)$ EXISTS IF AND ONLY IF BOTH $\lim_{x \to x_0^-} f(x)$ AND $\lim_{x \to x_0^-} f(x)$ EXIST AND ARE EQUAL.

$x \to x_0^-$

$x \to x_0^+$

FOR $z = f(x,y)$ THE DEFINITION LOOKS ESSENTIALLY THE SAME :

GIVEN $z = f(x,y)$ AND A POINT $(x_0, y_0)$ IN THE PLANE WITH $f$ DEFINED AT LEAST "NEAR" $(x_0, y_0)$

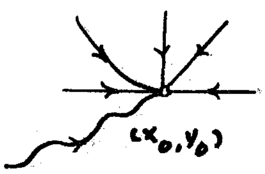



WE SAY THAT

$$\lim_{(x,y) \to (x_0, y_0)} f(x,y) = L$$

IF $f(x,y)$ CAN BE MADE AS CLOSE AS WE LIKE TO $L$ BY CHOOSING $(x,y)$ SUFFICIENTLY CLOSE (BUT NOT EQUAL) TO $(x_0, y_0)$.

THIS TIME, HOWEVER, INSTEAD OF JUST TWO THERE INFINITELY MANY "APPROACHES" TO $(x_0, y_0)$ AND, IN ORDER FOR THE LIMIT TO EXIST, THEY MUST ALL GIVE THE SAME RESULT.



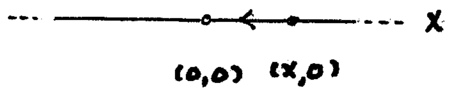

LET'S SEE HOW EASILY THIS CAN FAIL :

**EXAMPLE :** $z = f(x,y) = \dfrac{xy}{x^2+y^2}$

DOMAIN : ALL $(x,y) \neq (0,0)$

WE WILL CONSIDER THE LIMIT OF THIS FUNCTION AS $(x,y) \to (0,0)$.

APPROACH ALONG X-AXIS $(y=0)$ :




$$(x,y) = (x,0) \Rightarrow f(x,y) = f(x,0)$$

$$= \frac{x \cdot 0}{x^2 + 0^2}$$

$$= \frac{0}{x^2}$$

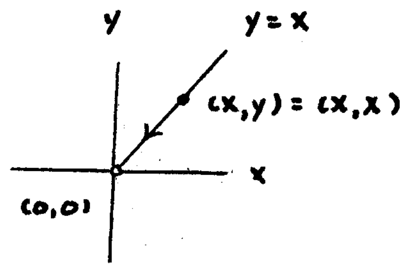
$$= 0$$

THE FUNCTION IS 0 AT EACH POINT OF THE X-AXIS

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{ALONG } y=0}} f(x,y) = \lim_{x \to 0} 0 = 0$$

WE GET THE SAME RESULT IF WE APPROACH ALONG THE y-AXIS $(x=0)$.

HOWEVER, SUPPOSE WE APPROACH ALONG THE LINE $y = x$ :



$(x,y) = (x,x) \Rightarrow f(x,y) = f(x,x) = \dfrac{x \cdot x}{x^2+x^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2}.$

THE FUNCTION TAKES THE VALUE $\frac{1}{2}$ AT EACH POINT OF $y = x$ $(x \neq 0)$

SO

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{ALONG } y=x}} f(x,y) = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

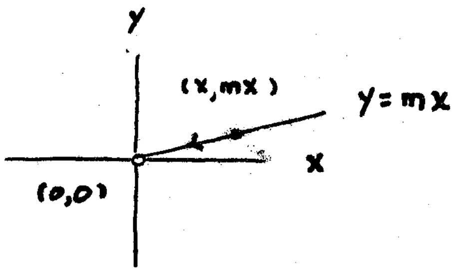
THUS, TWO DIFFERENT "APPROACHES" GIVE DIFFERENT RESULTS SO

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

<u>DOES</u> <u>NOT</u> <u>EXIST</u>.

<u>NOTE</u> : THE SITUATION HERE IS EVEN STRANGER THAN IT SEEMS IN THAT THE LIMIT OF THIS FUNCTION AS $(x,y) \to (0,0)$ IS <u>DIFFERENT</u> <u>ALONG</u> <u>EVERY</u> <u>STRAIGHT</u> <u>LINE</u> $y = mx$ THROUGH THE ORIGIN :

$$(x,y) = (x,mx) \Rightarrow f(x,y) = f(x,mx) = \frac{x(mx)}{x^2+(mx)^2}$$




$$= \frac{mx^2}{x^2+m^2x^2}$$

$$= \frac{mx^2}{(1+m^2)x^2}$$

$$= \frac{m}{1+m^2} \quad \text{SO}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{ALONG } y=mx}} f(x,y) = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$ WHICH IS DIFFERENT FOR DIFFERENT $m$ .

**ANOTHER EXAMPLE** : $z = f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$

DOMAIN : ALL $(x,y) \neq (0,0)$

AGAIN WE LOOK AT WHAT HAPPENS TO $f(x,y)$ AS $(x,y) \to (0,0)$.

NOTICE THAT THIS TIME ALONG $y = mx$,

$$f(x,y) = f(x,mx) = \frac{x^2(mx)}{x^4 + (mx)^2} = \frac{mx^3}{x^4 + m^2x^2}$$

$$= \frac{mx^3}{x^2(x^2+m^2)}$$

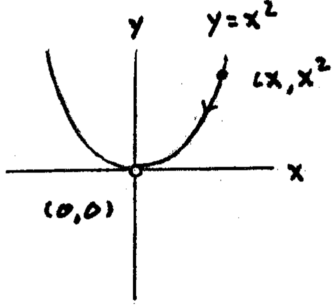
$$= \frac{mx}{x^2+m^2}$$

SO

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{ALONG } y = mx}} f(x,y) = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$$

THUS, ALL OF THESE APPROACHES GIVE THE SAME RESULT.

HOWEVER, LET'S NOW APPROACH ALONG $y = x^2$ :




$$\lim_{\substack{(x,y)\to(0,0) \\ \text{ALONG } y = x^2}} f(x,y) = \lim_{x\to 0} f(x,x^2)$$

$$= \lim_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2}$$

$$= \lim_{x\to 0} \frac{x^4}{x^4+x^4}$$

$$= \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

SO, AGAIN, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ **DOES NOT EXIST**.

<u>ONE</u> <u>LAST</u> <u>EXAMPLE</u> : $z = f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$

DOMAIN : ALL $(x,y) \neq (0,0)$

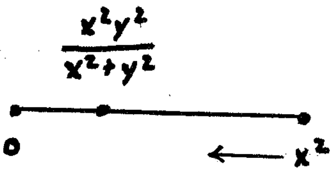
THIS TIME SOMETHING VERY DIFFERENT HAPPENS BECAUSE

$$0 \le \frac{x^2y^2}{x^2+y^2}$$

$$0 \le \frac{x^2y^2}{x^2+y^2} = x^2 \underbrace{\left( \frac{y^2}{x^2+y^2} \right)}_{\le 1} \le x^2$$

$$0 \le \frac{x^2y^2}{x^2+y^2} \le x^2$$

AS $(x,y) \to (0,0)$ , $x \to 0$ AND SO $x^2 \to 0$. THUS, $\dfrac{x^2y^2}{x^2+y^2}$ IS TRAPPED BETWEEN $0$ AND SOMETHING APPROACHING $0$




SO IT MUST ALSO APPROACH $0$ ( "SQUEEZE THEOREM" ), I.E.,

$$\lim_{(x,y) \to (0,0)} \frac{x^2y^2}{x^2+y^2} = 0.$$

<u>BOTTOM</u> <u>LINE</u> : LIMITS FOR FUNCTIONS OF TWO VARIABLES ARE TRICKY (AND IT'S WORSE FOR FUNCTIONS OF THREE OR MORE VARIABLES ).

OF COURSE, SOME LIMITS ARE JUST AS EASY FOR FUNCTIONS OF TWO VARIABLES AS THE ARE FOR FUNCTIONS OF ONE VARIABLE (FOR EXAMPLE, THOSE THAT ARE <u>NOT</u> <u>INDETERMINATE</u>), E.G.,

1. $$\lim_{(x,y)\to(1,2)} (x^2y + y^3) = 1^2\cdot 2 + 2^3 = 10$$ (POLYNOMIALS)

2. $$\lim_{(x,y)\to(1,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} \begin{array}{l} \to 1-(1)(1)+3 = 3 \\ \to 1^2\cdot 1 + 5(1)(1) - 1^3 = 5 \end{array}$$

$$= \frac{3}{5}$$ (RATIONAL FUNCTIONS FOR WHICH THE DENOMINATOR DOES NOT APPROACH 0)

3. $$\lim_{(x,y)\to(1,1)} \frac{1}{x-y} \begin{array}{l} \to 1 \\ \to 0 \end{array}$$ SO THE LIMIT DOES NOT EXIST

4. $$\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} \begin{array}{l} \to 0 \\ \to 0 \end{array}$$

$$= \lim_{(x,y)\to(1,1)} \frac{(x-y)(x+y)}{x-y}$$

$$= \lim_{(x,y)\to(1,1)} (x+y)$$ (PROVIDED $y \neq x$)

$$= 1+1 = 2$$

**RECALL:** $f(x)$ IS <u>CONTINUOUS AT</u> $x_0$ IF $\lim_{x \to x_0} f(x) = f(x_0)$.

IMPLICIT IN THIS IS

1. $x_0$ IS IN THE DOMAIN OF $f(x)$ SO $f(x_0)$ EXISTS,
2. $\lim_{x \to x_0} f(x)$ EXISTS,
3. THESE TWO ARE THE SAME.

FOR FUNCTIONS OF TWO VARIABLES THE DEFINITION IS THE SAME.

$f(x,y)$ IS <u>CONTINUOUS AT</u> $(x_0, y_0)$ IF

$$\lim_{(x,y) \to (x_0, y_0)} f(x,y) = f(x_0, y_0).$$

IF THIS IS TRUE FOR EVERY $(x_0, y_0)$ IN THE DOMAIN OF $f(x,y)$ WE SAY SIMPLY THAT $f(x,y)$ IS <u>CONTINUOUS</u>.

- POLYNOMIALS ARE CONTINUOUS EVERYWHERE.
- RATIONAL FUNCTIONS ARE CONTINUOUS WHEREVER THE DENOMINATOR IS NONZERO.
- SUMS, DIFFERENCES, AND PRODUCTS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS.

- QUOTIENTS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS WHEREVER THE DENOMINATOR IS NONZERO.

- IF $f(x,y)$ IS CONTINUOUS AND $g(u)$ IS A CONTINUOUS FUNCTION OF ONE VARIABLE, THEN $g(f(x,y))$ IS CONTINUOUS, E.G.,

$$\sin(5x^2y^3)$$

IS CONTINUOUS.

EXAMPLES :

1. $f(x,y) = \dfrac{xy}{x^2+y^2}$ IS NOT CONTINUOUS AT $(0,0)$ SINCE IT IS NOT DEFINED THERE.

2. $g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ 1 & \text{IF } (x,y) = (0,0) \end{cases}$ IS NOT CONTINUOUS AT $(0,0)$ SINCE $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ DOES NOT EXIST.

3. $$h(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ 1 & \text{IF } (x,y) = (0,0) \end{cases}$$ IS NOT

CONTINUOUS AT $(0,0)$ SINCE

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$$

$$= 0 \neq 1 = h(0,0)$$

4. $$H(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{IF } (x,y) = (0,0) \end{cases}$$

<u>IS</u> CONTINUOUS AT $(0,0)$

ALL OF THIS EXTENDS EASILY TO FUNCTIONS OF THREE VARIABLES.