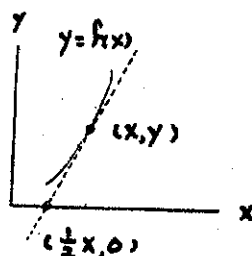


INTRODUCTION TO DIFFERENTIAL EQUATIONS :

BEGIN WITH SOME MOTIVATION.

1. CONSIDER THE FOLLOWING GEOMETRICAL PROBLEM :

FIND A FUNCTION $y = f(x)$ ON $x > 0$
WHOSE TANGENT LINE AT ANY POINT (x, y)
INTERSECTS THE x -AXIS AT $(\frac{1}{2}x, 0)$



COMPUTE THE SLOPE OF THE TANGENT LINE AT (x, y) IN TWO WAYS :

$$\frac{dy}{dx} = \frac{y - 0}{x - \frac{1}{2}x}$$

$$\boxed{\frac{dy}{dx} = \frac{2y}{x}}$$

NOTE THAT THIS DOES NOT TELL US WHAT $y = f(x)$ IS, BUT ONLY TELLS US WHAT CONDITION IT MUST SATISFY IN ORDER TO BE A SOLUTION TO OUR PROBLEM.

2. RADIOACTIVE SUBSTANCES (AT LEAST UNDER A CERTAIN RANGE OF CONDITIONS) DECAY AT A RATE PROPORTIONAL TO THE AMOUNT OF THE SUBSTANCE PRESENT.

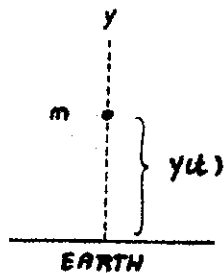
$$Q(t) = \text{AMOUNT PRESENT AT TIME } t$$

THEN FOR SOME CONSTANT $k < 0$,

$$\frac{dQ}{dt} = kQ$$

SAME STORY AS IN # 1. THIS DOESN'T TELL US WHAT $Q(t)$ IS, BUT ONLY GIVES US A CONDITION $Q(t)$ MUST SATISFY WHICH AGAIN INVOLVES THE DERIVATIVE OF THE UNKNOWN FUNCTION $Q(t)$.

3. LAST TERM WE SAW THAT AN OBJECT FALLING FREELY IN THE EARTH'S GRAVITATIONAL FIELD EXPERIENCES A CONSTANT DOWNWARD ACCELERATION.



$$m \frac{d^2y}{dt^2} = -mg \quad (\text{NEWTON'S 2}^{\text{ND}} \text{ LAW})$$

OR

$$\frac{dv}{dt} = -g \quad (v = \frac{dy}{dt})$$

IF IT ISN'T FALLING "FREELY" BUT IS SUBJECT TO AIR RESISTANCE (GENERALLY ASSUMED PROPORTIONAL TO THE VELOCITY), THEN

$$m \frac{dv}{dt} = -mg - cv$$

FOR SOME CONSTANT $c > 0$.

$$\frac{dv}{dt} + \frac{c}{m}v = -g$$

ONCE AGAIN WE HAVE AN EQUATION FOR THE UNKNOWN FUNCTION $v(t)$ INVOLVING THAT FUNCTION AND ITS DERIVATIVE.

IF WE WROTE THIS IN TERMS OF THE HEIGHT $y(t)$ IT WOULD READ

$$\frac{d^2y}{dt^2} + \frac{c}{m} \frac{dy}{dt} = -g$$

AN ORDINARY DIFFERENTIAL EQUATION (ODE) IS AN EQUATION TO BE SOLVED FOR SOME UNKNOWN FUNCTION THAT INVOLVES THAT FUNCTION AND ITS DERIVATIVES.

THE ORDER OF THE ODE IS THE ORDER OF THE HIGHEST ORDER DERIVATIVE THAT APPEARS IN THE EQUATION.

E.G., $\frac{dy}{dx} = \frac{y}{x}$ AND $\frac{dq}{dt} = kq$ ARE

1ST ORDER, BUT $\frac{d^2y}{dt^2} + c \frac{dy}{dt} = -g$ IS 2ND ORDER

A SOLUTION TO AN ODE IS JUST A FUNCTION THAT SATISFIES
THE EQUATION, E.G.,

$$1. \quad y' = y$$

SOLUTIONS :

$$y = e^x$$

$$y = 3e^x$$

$$\vdots$$

$$y = Ae^x \quad \text{FOR ANY CONSTANT } A$$

$$2. \quad y'' + y = 0$$

SOLUTIONS :

$$y = \sin x$$

$$y = \cos x$$

$$y = \sin x + \cos x$$

$$y = A \sin x \quad \text{FOR ANY CONSTANT } A$$

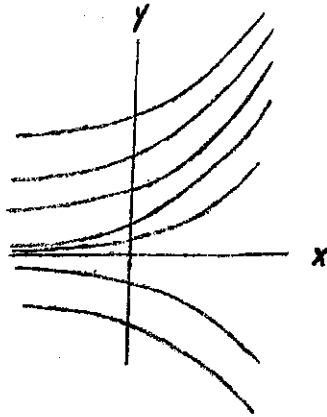
$$y = B \cos x \quad \text{FOR ANY CONSTANT } B$$

$$y = A \sin x + B \cos x$$

GENERALLY, A DIFFERENTIAL EQUATION HAS INFINITELY MANY
SOLUTIONS, BUT IT MIGHT ONLY HAVE ONE ($(y')^2 + y^2 = 0$)
OR IT MIGHT HAVE NONE ($(y')^2 + y^2 = -1$).

THE GRAPHS OF THE SOLUTIONS TO A DIFFERENTIAL EQUATION ARE CALLED ITS INTEGRAL CURVES.

E.G., THE INTEGRAL CURVES OF $y' = y$ ARE



$y = Ae^x$ FOR VARIOUS
CONSTANTS A

NOTE: FOR THIS 1ST ORDER EQUATION THERE IS EXACTLY ONE INTEGRAL CURVE THROUGH EVERY POINT (x_0, y_0) IN THE PLANE.

GENERAL 1ST ORDER ODE: $y' = f(x, y)$

E.G., $y' = y$, $y' = x^2 + y^2$

PROBLEM: FIND ALL THE SOLUTIONS.

GENERAL 1ST ORDER INITIAL VALUE PROBLEM: $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$

PROBLEM: FIND THE SOLUTION TO $y' = f(x, y)$ WHOSE GRAPH GOES THROUGH THE POINT (x_0, y_0) .

MOST OF THESE PROBLEMS ARE EXTREMELY DIFFICULT. WE WILL CONSIDER JUST THREE SPECIAL CASES THAT ARE MANAGEABLE AND ACTUALLY ARISE OFTEN IN APPLICATIONS.

1. ("SIMPLE" INTEGRATION)

$$\frac{dy}{dx} = f(x)$$

NO y ON THE RIGHT HAND SIDE. JUST INTEGRATE BOTH SIDES WITH RESPECT TO x .

EXAMPLES :

1. SOLVE : $\frac{dy}{dx} = x\sqrt{x^2+4}$

$$\begin{aligned} \text{INTEGRATE : } y &= \int x\sqrt{x^2+4} dx \\ &= \frac{1}{2} \int (x^2+4)^{\frac{1}{2}} (2x dx) \\ &= \frac{1}{2} \cdot \frac{2}{3} (x^2+4)^{\frac{3}{2}} + C \\ y &= \frac{1}{3} (x^2+4)^{\frac{3}{2}} + C \end{aligned}$$

2. SOLVE THE INITIAL VALUE PROBLEM : $\left\{ \begin{array}{l} \frac{dy}{dx} = x\sqrt{x^2+4} \\ y(-4) = 0 \end{array} \right.$

IN THE FIRST PROBLEM WE FOUND ALL OF THE SOLUTIONS TO

$$\frac{dy}{dx} = x\sqrt{x^2+4} :$$

$$y(x) = \frac{1}{3}(x^2+4)^{\frac{3}{2}} + C$$

NOW WE SOLVE FOR THE C THAT WILL ENSURE

$$0 = y(-4) = \frac{1}{3}((-4)^2+4)^{\frac{3}{2}} + C$$

$$0 = \frac{1}{3}(20)^{\frac{3}{2}} + C$$

$$0 = \frac{1}{3} \cdot 20\sqrt{20} + C$$

$$0 = \frac{40\sqrt{5}}{3} + C$$

$$C = -\frac{40\sqrt{5}}{3}$$

$$y(x) = \frac{1}{3}(x^2+4)^{\frac{3}{2}} - \frac{40\sqrt{5}}{3}$$

2. (LINEAR EQUATIONS)

A 1ST ORDER LINEAR ODE IS ONE THAT CAN BE WRITTEN IN THE FORM

$$y' + P(x)y = Q(x)$$

FOR SOME (KNOWN) FUNCTIONS $P(x)$ AND $Q(x)$, E.G.,

$$y' + 3y = e^{-2x}$$

$$\frac{dN}{dt} + \frac{c}{m} N = -g$$

$$\frac{dQ}{dt} = kQ \quad \left(\frac{dQ}{dt} - kQ = 0 \right)$$

$$\frac{dy}{dx} = \frac{2y}{x} \quad \left(\frac{dy}{dx} - \frac{2}{x} y = 0 \right)$$

NOTE THAT YOU CAN'T SOLVE THESE BY JUST "INTEGRATING BOTH SIDES" SINCE THIS WOULD BURY THE UNKNOWN FUNCTION INSIDE AN INTEGRAL SIGN.

THERE IS, HOWEVER, A "TRICK" WHICH GETS AROUND THIS PROBLEM.

CONSIDER

$$y' + 3y = e^{-2x}$$

LOOK WHAT HAPPENS WHEN YOU MULTIPLY BOTH SIDES BY

$$\mu(x) = e^{3x}$$

(I'LL EXPLAIN WHERE THIS CAME FROM SHORTLY)

$$e^{3x} y' + 3e^{3x} y = e^x$$



PRODUCT RULE FOR

$$(e^{3x} y)' = e^x$$

NOW WE CAN INTEGRATE BOTH SIDES WITHOUT BURYING y INSIDE AN INTEGRAL SIGN :

$$\int (e^{3x} y)' dx = \int e^x dx$$

$$e^{3x} y = e^x + C$$

$$y = e^{-2x} + ce^{-3x}$$

SO THE QUESTION IS, "WHERE DID THE MAGICAL $\mu(x) = e^{3x}$ COME FROM ?"

NOTE : $y' + 3y = e^{-2x}$

$$y' + P(x)y = Q(x)$$

HERE $P(x) = 3$ SO $\mu(x) = e^{\int P(x) dx}$

(JUST ONE ANTIDERIVATIVE NEEDED IN THE EXPONENT)

WE WILL NOW SHOW THAT FOR ANY LINEAR EQUATION

$$y' + P(x)y = Q(x)$$

THE FUNCTION

$$\mu(x) = e^{\int P(x) dx}$$

(CALLED AN INTEGRATING FACTOR FOR THE EQUATION) WORKS JUST

THE WAY e^{3x} DID IN THE EXAMPLE, I.E., MULTIPLYING BOTH SIDES OF $y' + P(x)y = Q(x)$ BY $\mu(x)$ TURNS THE EQUATION INTO

$$(\mu(x)y)' = \mu(x)Q(x)$$

WHICH CAN JUST BE INTEGRATED AND SOLVED FOR y .

HERE'S THE REASON:

$$y' + P(x)y = Q(x)$$

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

$$(\mu(x)y)' = \mu(x)y' + \mu'(x)y$$

$$= \mu(x)y' + (e^{\int P(x)dx})'y$$

$$= \mu(x)y' + e^{\int P(x)dx} (\int P(x)dx)'y$$

$$= \mu(x)y' + \mu(x)P(x)y$$

$$(\mu(x)y)' = \mu(x)Q(x)$$

EXAMPLES:

1. SOLVE : $y' - 2xy = x$

HERE $P(x) = -2x$ SO

$$\mu(x) = e^{\int P(x) dx} = e^{\int -2x dx} = e^{-x^2}$$

SO

$$e^{-x^2} [y' - 2xy = x]$$

$$e^{-x^2} y' - 2xe^{-x^2} y = xe^{-x^2}$$

$$(e^{-x^2} y)' = xe^{-x^2}$$

$$e^{-x^2} y = \int xe^{-x^2} dx$$

$$e^{-x^2} y = -\frac{1}{2} \int e^{-x^2} (-2x dx)$$

$$e^{-x^2} y = -\frac{1}{2} e^{-x^2} + C$$

$$y = -\frac{1}{2} + ce^{x^2}$$

2. SOLVE

$$\begin{cases} x^2 y' + xy = x \sin x & \text{ON } x > 0 \\ y(\pi) = 2 \end{cases}$$

ON $x > 0$ WE CAN DIVIDE THE EQUATION BY x^2 TO GET

$$y' + \frac{1}{x} y = \frac{\sin x}{x}$$

SO

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = e^{\ln x} = x$$

$$x [y' + \frac{1}{x} y = \frac{\sin x}{x}]$$

$$xy' + y = \sin x$$

$$(xy)' = \sin x$$

$$xy = \int \sin x \, dx$$

$$xy = -\cos x + C$$

$$y = -\frac{\cos x}{x} + \frac{C}{x}$$

TO ENSURE $y(\pi) = 2$,

$$2 = y(\pi) = -\frac{\cos \pi}{\pi} + \frac{C}{\pi}$$

$$2 = \frac{1}{\pi} + \frac{C}{\pi}$$

$$2\pi = 1 + C$$

$$C = 2\pi - 1$$

SO

$$y = -\frac{\cos x}{x} + \frac{2\pi - 1}{x}$$

3. SOLVE $\begin{cases} \frac{dv}{dt} + \frac{c}{m}v = -g \\ v(0) = v_0 \end{cases}$ AND INVESTIGATE $\lim_{t \rightarrow \infty} v(t)$.

$$\mu(t) = e^{\int \frac{c}{m} dt} = e^{\frac{c}{m}t}$$

$$e^{\frac{c}{m}t} \left[\frac{dv}{dt} + \frac{c}{m}v = -g \right]$$

$$e^{\frac{c}{m}t} \frac{dv}{dt} + \frac{c}{m} e^{\frac{c}{m}t} v = -g e^{\frac{c}{m}t}$$

$$(e^{\frac{c}{m}t} v)' = -g e^{\frac{c}{m}t}$$

$$e^{\frac{c}{m}t} v = \int -g e^{\frac{c}{m}t} dt$$

$$e^{\frac{c}{m}t} v = -\frac{gm}{c} e^{\frac{c}{m}t} + D$$

("c" IS ALREADY IN USE)

$$v(t) = -\frac{gm}{c} + D e^{-\frac{c}{m}t}$$

$$v_0 = v(0) = -\frac{gm}{c} + D$$

$$v_0 + \frac{gm}{c} = D$$

THUS,

$$v(t) = -\frac{gm}{c} + \left(v_0 + \frac{gm}{c}\right) e^{-\frac{c}{m}t}$$

SINCE c AND m ARE POSITIVE CONSTANTS,

$$\lim_{t \rightarrow \infty} v(t) = -\frac{gm}{c}$$

= TERMINAL VELOCITY

3. (SEPARABLE EQUATIONS)

A 1ST ORDER SEPARABLE ODE IS ONE THAT CAN BE WRITTEN
IN THE FORM

$$h(y) \frac{dy}{dx} = g(x)$$

OR, IN "DIFFERENTIAL FORM"

$$h(y) dy = g(x) dx$$

(VARIABLES "SEPARATED" ON OPPOSITE SIDES OF THE EQUATION)

E.G.,

$$\frac{dy}{dx} = \frac{2x^2y + y}{x} \quad \left(\frac{dy}{dx} = \frac{(2x^2+1)y}{x} \Rightarrow \right.$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2x^2+1}{x} \Rightarrow$$

$$\left. \frac{1}{y} dy = \frac{2x^2+1}{x} dx \right)$$

$$\frac{dy}{dx} = \frac{2y}{x} \quad \left(\frac{1}{y} \frac{dy}{dx} = \frac{2}{x} \Rightarrow \frac{1}{y} dy = \frac{2}{x} dx \right)$$

$$\frac{dQ}{dt} = kQ \quad \left(\frac{1}{Q} \frac{dQ}{dt} = k \Rightarrow \frac{1}{Q} dQ = k dt \right)$$

TO SEE HOW SUCH EQUATIONS CAN BE SOLVED :

$$h(y) \frac{dy}{dx} = g(x)$$

LET $H(y)$ AND $G(x)$ BE ANTIDERIVATIVES FOR $h(y)$ AND $g(x)$.

$$\frac{dH}{dy} = h(y) \quad \frac{dG}{dx} = g(x)$$

EQUATION BECOMES

$$\frac{dH}{dy} \frac{dy}{dx} = \frac{dG}{dx}$$

NOW RECALL THE CHAIN RULE :

$$H = H(y) \quad \text{AND} \quad y = y(x) \Rightarrow$$

$$H = H(y(x))$$

$$\frac{dH}{dx} = \frac{dH}{dy} \frac{dy}{dx}$$

THUS, THE EQUATION SAYS

$$\frac{d}{dx} H(y) = \frac{d}{dx} G(x)$$

INTEGRATE BOTH SIDES WITH RESPECT TO X :

$$H(y) = G(x) + C$$

NOTICE THAT THIS IS EXACTLY WHAT YOU WOULD GET BY FORMALLY
INTEGRATING BOTH SIDES OF

$$h(y)dy = g(x)dx$$

$$\int h(y)dy = \int g(x)dx$$

$$H(y) = G(x) + C$$

AS A PRACTICAL MATTER THIS IS HOW THE EQUATIONS ARE
ACTUALLY SOLVED.

EXAMPLES :

$$\begin{aligned}
 1. \quad \frac{dy}{dx} &= 1+x^2+y^2+x^2y^2 \\
 &= (1+x^2) + y^2(1+x^2) \\
 &= (1+x^2)(1+y^2)
 \end{aligned}$$

$$\frac{1}{1+y^2} \frac{dy}{dx} = 1+x^2$$

$$\frac{1}{1+y^2} dy = (1+x^2) dx$$

$$\int \frac{1}{1+y^2} dy = \int (1+x^2) dx$$

$$\arctan y = x + \frac{1}{3}x^3 + C \quad (\text{IMPLICIT SOLUTION})$$

$$y = \tan(x + \frac{1}{3}x^3 + C) \quad (\text{EXPLICIT SOLUTION})$$

$$2. \quad \frac{dy}{dx} = \frac{2x^2y + y}{x} = \frac{y(2x^2+1)}{x} \quad (x \neq 0)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2x^2+1}{x}$$

NOTE : THIS ASSUMES $y \neq 0$. HOWEVER,
 $y = 0$ IS A SOLUTION TO THE EQUATION.

$$\frac{1}{y} dy = (2x + \frac{1}{x}) dx$$

$$\int \frac{1}{y} dy = \int (2x + \frac{1}{x}) dx$$

$$\ln|y| = x^2 + \ln|x| + C$$

$$\ln|y| - \ln|x| = x^2 + c$$

$$\ln\left|\frac{y}{x}\right| = x^2 + c$$

$$\left|\frac{y}{x}\right| = e^{x^2+c} = e^c e^{x^2}$$

$$\frac{y}{x} = (\pm e^c) e^{x^2}$$

$$\frac{y}{x} = k e^{x^2}$$

$$y = k x e^{x^2}$$

NOTE : $k = \pm e^c$ IS AN ARBITRARY NONZERO CONSTANT. HOWEVER, TAKING $k = 0$ GIVES THE SOLUTION $y = 0$ THAT WE ELIMINATED EARLIER.