

INTEGRATION BY PARTS

RECALL THE PRODUCT RULE FOR DERIVATIVES: $u = u(x)$, $v = v(x)$

$$\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$u \frac{dv}{dx} = \frac{d}{dx} (uv) - v \frac{du}{dx}$$

INTEGRATE BOTH SIDES:

$$\int u \frac{dv}{dx} dx = \int \frac{d}{dx} (uv) dx - \int v \frac{du}{dx} dx$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

SHORTHAND NOTATION: THE INTEGRATION BY PARTS FORMULA:

$$\int u dv = uv - \int v du$$

EXAMPLES:

1. $\int x \sin x dx$: IDENTIFY SOMETHING TO CALL u . THE REST IS dv .
COMPUTE du BY DIFFERENTIATION AND v BY
INTEGRATION. PLUG IN THE INTEGRATION BY PARTS
FORMULA AND HOPE THAT THE NEW INTEGRAL IS
EASIER THAN THE ORIGINAL ONE.

GENERALLY TRY TO CHOOSE u TO BE SOMETHING THAT
SIMPLIFIES WHEN YOU DIFFERENTIATE IT.

$$\int x \sin x dx = (x)(-\cos x) - \int (-\cos x) dx$$

$$u = x \quad dv = \sin x dx \quad = -x \cos x + \int \cos x dx$$

$$du = dx \quad v = -\cos x \quad = -x \cos x + \sin x + C$$

$$2. \int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$$

$$u = x \quad dv = e^x dx$$

$$du = dx \quad v = e^x$$

$$3. \int x^2 e^{-x} dx = (x^2)(-e^{-x}) - \int (-e^{-x})(2x dx)$$

$$u = x^2 \quad dv = e^{-x} dx$$

$$du = 2x dx \quad v = -e^{-x}$$

$$= -x^2 e^{-x} + 2 \int x e^{-x} dx$$

$$u = x \quad dv = e^{-x} dx$$

$$du = dx \quad v = -e^{-x}$$

$$= -x^2 e^{-x} + 2 [-x e^{-x} - \int -e^{-x} dx]$$

$$= -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} dx$$

$$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

$$4. \int \ln x \, dx$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \left(\frac{1}{x} dx \right) = x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \end{aligned}$$

$$5. \int \operatorname{arctan} x \, dx$$

$$u = \operatorname{arctan} x \quad dv = dx$$

$$du = \frac{1}{1+x^2} dx \quad v = x$$

$$\int \operatorname{arctan} x \, dx = x \operatorname{arctan} x - \int x \left(\frac{1}{1+x^2} dx \right)$$

$$= x \operatorname{arctan} x - \int \frac{1}{1+x^2} x \, dx$$

$$= x \operatorname{arctan} x - \frac{1}{2} \int \frac{1}{1+x^2} (2x \, dx)$$

$$u = 1+x^2 \quad (\text{DIFFERENT } u!)$$

$$du = 2x \, dx$$

$$= x \operatorname{arctan} x - \frac{1}{2} \int \frac{1}{u} \, du$$

$$= x \operatorname{arctan} x - \frac{1}{2} \ln |u| + C$$

$$= x \operatorname{arctan} x - \frac{1}{2} \ln |1+x^2| + C$$

$$= x \operatorname{arctan} x - \frac{1}{2} \ln (1+x^2) + C$$

$$6. \int_{\sqrt{e}}^e \frac{\ln x}{x^2} dx = \int_{\sqrt{e}}^e (\ln x) x^{-2} dx$$

$$u = \ln x \quad dv = x^{-3} dx$$

$$du = \frac{1}{x} dx \quad v = -\frac{1}{2} x^{-2}$$

$$= -\frac{1}{2} x^{-2} \ln x \Big|_{\sqrt{e}}^e - \int_{\sqrt{e}}^e (-\frac{1}{2} x^{-2}) (x^{-1} dx)$$

$$= -\frac{1}{2} \frac{\ln x}{x^2} \Big|_{\sqrt{e}}^e + \frac{1}{2} \int_{\sqrt{e}}^e x^{-3} dx$$

$$= -\frac{1}{2} \left[\frac{\ln e}{e^2} - \frac{\ln \sqrt{e}}{(\sqrt{e})^2} \right] + \frac{1}{2} \left(-\frac{1}{2} x^{-2} \Big|_{\sqrt{e}}^e \right)$$

$$= -\frac{1}{2} \left[\frac{1}{e^2} - \frac{\frac{1}{2}}{e} \right] - \frac{1}{4} \left[\frac{1}{e^2} - \frac{1}{e} \right]$$

$$= -\frac{1}{2} \left(\frac{1}{e^2} \right) + \frac{1}{4} \left(\frac{1}{e} \right) - \frac{1}{4} \left(\frac{1}{e^2} \right) + \frac{1}{4} \left(\frac{1}{e} \right)$$

$$= -\frac{3}{4} \left(\frac{1}{e^2} \right) + \frac{1}{2} \left(\frac{1}{e} \right)$$

$$7. \int \sin \sqrt{x} dx$$

THIS ONE IS TRICKY. FIRST MAKE THE (NONOBVIOUS) SUBSTITUTION

$t = \sqrt{x}$ so $dt = \frac{1}{2\sqrt{x}} dx$, I.E., $dx = 2\sqrt{x} dt = 2t dt$. THEN

$$\begin{aligned} \int \sin \sqrt{x} dx &= \int \sin t (2t dt) \\ &= 2 \int t \sin t dt \end{aligned}$$

NOW, INTEGRATION BY PARTS:

$$\int \sin \sqrt{x} dx = 2 \int t \sin t dt$$

$$u = t \quad dv = \sin t dt$$

$$du = dt \quad v = -\cos t$$

$$= 2 (-t \cos t + \int \cos t dt)$$

$$= -2t \cos t + 2 \sin t + C$$

$$= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + C$$

8. $\int e^x \cos x dx$

NO OBVIOUS CHOICE FOR u (SOMETHING THAT SIMPLIFIES WHEN YOU DIFFERENTIATE IT) THIS TIME. JUST MAKE SOME CHOICE, E.G.,

$$u = e^x \quad dv = \cos x dx$$

$$du = e^x dx \quad v = \sin x$$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

DOESN'T SEEM TO HAVE HELPED MUCH.
JUST SWAPPED A COS FOR A SIN.

HOWEVER, WATCH WHAT HAPPENS WHEN THE INTEGRATION BY PARTS FORMULA IS APPLIED TO THE NEW INTEGRAL.

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$u = e^x \quad dv = \sin x dx$$

$$du = e^x dx \quad v = -\cos x$$

$$= e^x \sin x - (-e^x \cos x + \int e^x \cos x dx)$$

$$\int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$+ \int e^x \cos x dx \quad + \int e^x \cos x dx$$

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x dx = \frac{1}{2} (e^x \sin x + e^x \cos x) + C$$

9. PROVE THE REDUCTION FORMULA : FOR $n \geq 2$,

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \int \cos^{n-1} x \cos x dx$$

$$u = \cos^{n-1} x$$

$$dv = \cos x dx$$

$$du = (n-1) \cos^{n-2} x (-\sin x) dx$$

$$v = \sin x$$

$$\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx$$

$$\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$+ (n-1) \int \cos^n x dx$$

$$+ (n-1) \int \cos^n x dx$$

$$(1 + (n-1)) \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$\underbrace{\hspace{1.5cm}}_n$

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

AS REQUIRED.

10. USE THE REDUCTION FORMULA TO FIND $\int \cos^4 x dx$

$$\int \cos^4 x dx = \frac{1}{4} \cos^{4-1} x \sin x + \frac{4-1}{4} \int \cos^{4-2} x dx$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x dx$$

$\underbrace{\hspace{1.5cm}}$

REDUCTION FORMULA AGAIN!

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left[\frac{1}{2} \cos^{2-1} x \sin x + \frac{2-1}{2} \int \cos^{2-2} x dx \right]$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} \int 1 dx$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C$$