

INFINITE SERIES

LET'S BUILD A SEQUENCE $\{S_0, S_1, S_2, \dots\}$ IN THE FOLLOWING WAY :

$$S_0 = 1$$

$$S_1 = 1 + \frac{1}{2}$$

$$S_2 = 1 + \frac{1}{2} + \frac{1}{4}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

⋮

$$S_n = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^n$$

⋮

SUNS OF MORE AND MORE POWERS OF $\frac{1}{2}$.

QUESTION : DOES THE SEQUENCE CONVERGE ? THAT IS, AS WE ADD TOGETHER MORE AND MORE POWERS OF $\frac{1}{2}$, DO THE SUNS "APPROACH" ANYTHING ?

THE ANSWER WOULD CERTAINLY BE "NO" IF WE REPLACED THE $\frac{1}{2}$ WITH A 1 SINCE THE SEQUENCE WOULD CLEARLY BLOW UP (I.E., $\rightarrow \infty$).

HOWEVER, OUR SEQUENCE ACTUALLY DOES CONVERGE BECAUSE

1. IT IS INCREASING ($S_{n+1} = S_n + \text{A POSITIVE NUMBER}$).
2. BOUNDED FROM ABOVE BY 2 (GOING DOWN THE LIST FROM ONE TERM TO THE NEXT, WE ALWAYS ADD JUST HALF OF WHAT IT WOULD TAKE TO GET THE SUM UP TO 2).

THIS SHOULD BE A LITTLE SURPRIZING (ADDING MORE AND MORE POSITIVE NUMBERS, BUT THE SUMS DO NOT BLOW UP).

BUT WHAT DOES $\{ S_0, S_1, S_2, \dots \}$ CONVERGE TO ?

USUALLY ANSWERING QUESTIONS LIKE THIS IS VERY HARD, BUT FOR THIS ONE THERE IS A TRICK THAT PROVIDES THE ANSWER :

TEMPORARILY CALL THE LIMIT OF $\{ S_n \}$ AS $n \rightarrow \infty$ JUST S .

$$S = \lim_{n \rightarrow \infty} S_n$$

NOW NOTICE THAT

$$S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n$$

SO

$$\frac{1}{2} S_n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n+1}$$

SUBTRACT THESE TWO TO GET

$$S_n - \frac{1}{2} S_n = 1 - \left(\frac{1}{2}\right)^{n+1}$$

$$\frac{1}{2} S_n = 1 - \left(\frac{1}{2}\right)^{n+1}$$

$$S_n = 2 \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) = 2 - \frac{1}{2^n}$$

THUS,

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(2 - \frac{1}{2^n}\right) = 2$$

CONCLUSION : AS YOU ADD TOGETHER MORE AND MORE POWERS OF $\frac{1}{2}$ (STARTING WITH $\left(\frac{1}{2}\right)^0$) THE SUMS GET CLOSER AND CLOSER TO 2.

LET'S WRITE THIS OUT :

$$\lim_{n \rightarrow \infty} S_n = 2$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \left(\frac{1}{2}\right)^n\right) = 2$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \left(\frac{1}{2}\right)^k = 2$$

NOW WE INTRODUCE A SHORTHAND NOTATION FOR THIS :

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$$

OR SOMETIMES

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$$

PLEASE UNDERSTAND THAT NEITHER OF THESE LAST TWO IS REALLY A SUM (YOU CAN'T ADD UP INFINITELY MANY NUMBERS, PERIOD!)
 THEY ARE SHORTHANDS FOR LIMIT STATEMENTS.

A SYMBOL LIKE

$$\sum_{k=0}^{\infty} a_k$$

OR

$$a_0 + a_1 + a_2 + \dots$$

IS CALLED AN INFINITE SERIES (OR JUST SERIES). IT HAS A
 CORRESPONDING SEQUENCE OF PARTIAL SUMS

$$S_0 = a_0$$

$$S_1 = a_0 + a_1$$

$$S_2 = a_0 + a_1 + a_2$$

⋮

$$S_n = a_0 + a_1 + a_2 + \dots + a_n$$

⋮

IF THIS SEQUENCE $\{S_n\}$ OF PARTIAL SUMS CONVERGES TO SOME
 REAL NUMBER S , THEN THE SERIES IS SAID TO CONVERGE TO S
 AND S IS CALLED THE SUM OF THE SERIES. IN THIS CASE WE
 WRITE

$$\sum_{k=0}^{\infty} a_k = S$$

OR

$$a_0 + a_1 + a_2 + \dots = S.$$

IF THE SEQUENCE OF PARTIAL SUMS DOES NOT CONVERGE, THEN WE SAY THAT THE SERIES DIVERGES.

EXAMPLES:

1. THE SAME ARGUMENT GIVEN FOR $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$

PROVES THAT IF

$$\boxed{-1 < r < 1}$$

THEN

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

AND, IN FACT,

$$\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$$

(GEOMETRIC SERIES)

FOR ANY CONSTANT a , E.G.,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{3}{10^k} &= \sum_{k=0}^{\infty} 3\left(\frac{1}{10}\right)^k = \frac{3}{1-\frac{1}{10}} \\ &= \frac{3}{\frac{9}{10}} = \frac{10}{3} \end{aligned}$$

$$2. \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \dots \quad \text{DIVERGES BECAUSE ITS}$$

SEQUENCE OF PARTIAL SUMS IS

$$S_0 = 1$$

$$S_1 = 1 - 1 = 0$$

$$S_2 = 1 - 1 + 1 = 1$$

$$S_3 = 1 - 1 + 1 - 1 = 0$$

$$\vdots$$

AND THE SEQUENCE $\{1, 0, 1, 0, \dots\}$ DIVERGES.

NOTE : THIS IS THE GEOMETRIC SERIES ($a=1$)
WHEN $r = -1$. IN FACT, THE GEONERIC
SERIES DIVERGES WHENEVER $r \leq -1$ OR $r > 1$.

$$\begin{aligned}
 3. \sum_{k=0}^{\infty} 3^k 5^{1-k} &= \sum_{k=0}^{\infty} 3^k \cdot 5 \cdot 5^{-k} = \sum_{k=0}^{\infty} 5 \cdot 3^k \left(\frac{1}{5}\right)^k \\
 &= \sum_{k=0}^{\infty} 5 \left(\frac{3}{5}\right)^k = \frac{5}{1 - \frac{3}{5}} = \frac{5}{\frac{2}{5}} \\
 &= \frac{25}{2}
 \end{aligned}$$

$$4. \quad \sum_{k=1}^{\infty} \frac{5}{4^k} = \sum_{k=1}^{\infty} 5 \left(\frac{1}{4}\right)^k$$

NOTE THAT THE INDEX k STARTS AT 1 THIS TIME. WE KNOW THAT

$$\sum_{k=0}^{\infty} 5 \left(\frac{1}{4}\right)^k = \frac{5}{1 - \frac{1}{4}} = \frac{5}{\frac{3}{4}} = \frac{20}{3}$$

$$5 \left(\frac{1}{4}\right)^0 + \sum_{k=1}^{\infty} 5 \left(\frac{1}{4}\right)^k = \frac{20}{3} \quad (*)$$

$$\sum_{k=1}^{\infty} 5 \left(\frac{1}{4}\right)^k = \frac{20}{3} - 5 = \frac{5}{3}$$

SO

$$\sum_{k=1}^{\infty} \frac{5}{4^k} = \frac{5}{3}$$

NOTE: THE LINE WITH THE (*) ABOVE IS ACTUALLY A LITTLE SLOPPY. WHY? AND HOW DO YOU JUSTIFY IT?

5. THE FORMULA FOR THE SUM OF A GEOMETRIC SERIES CAN BE USED TO WRITE RATIONAL NUMBERS GIVEN AS DECIMALS (E.G., 0.784784784...) AS QUOTIENTS OF INTEGERS.

$$\begin{aligned} 0.784784784 \dots &= 0.784 + 0.000784 + 0.000000784 + \dots \\ &= 0.784(1) + 0.784(10^{-3}) + 0.784(10^{-6}) + \dots \\ &= 0.784 \left(\frac{1}{10^3}\right)^0 + 0.784 \left(\frac{1}{10^3}\right)^1 + 0.784 \left(\frac{1}{10^3}\right)^2 + \dots \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\infty} 0.784 \left(\frac{1}{10^3} \right)^k = \frac{0.784}{1 - \frac{1}{10^3}} \frac{10^3}{10^3} \\
 &= \frac{784}{999}
 \end{aligned}$$

GEOMETRIC SERIES ARE A RARE EXCEPTION TO A GENERAL RULE :
 OFTEN IT IS NOT SO HARD TO SHOW THAT A SERIES CONVERGES,
 BUT FINDING OUT WHAT IT CONVERGES TO IS EXTREMELY
 DIFFICULT .

THERE IS BASICALLY JUST ONE MORE EXCEPTION TO THIS RULE :

TELESCOPING SERIES :

CONSIDER THE SERIES $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$. WE'LL WRITE OUT
 ITS SEQUENCE OF PARTIAL SUMS :

$$S_1 = \frac{1}{1} - \frac{1}{2}$$

$$S_2 = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$S_3 = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right)$$

⋮

$$S_n = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

⋮

NOTICE THE CANCELING OF EVERY OTHER TERM . THE PARTIAL SUMS
 COLLAPSE (" TELESCOPE ") TO GIVE

$$S_n = 1 - \frac{1}{n+1}$$

AND THIS IS EASY TO TAKE THE LIMIT OF :

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \rightarrow \infty} S_n = 1.$$

ANOTHER EXAMPLE :
$$\sum_{k=0}^{\infty} \frac{1}{k^2+3k+2}$$

AT FIRST THIS DOESN'T LOOK AT ALL RIGHT AND JUST WRITING OUT THE PARTIAL SUMS DOESN'T LEAD TO ANY TELESCOPING. HOWEVER,

$$\frac{1}{k^2+3k+2} = \frac{1}{(k+1)(k+2)} = \frac{A}{k+1} + \frac{B}{k+2}$$

(PARTIAL FRACTIONS)

$$1 = A(k+2) + B(k+1)$$

$$k = -1 : 1 = A$$

$$k = -2 : 1 = -B$$

$$\frac{1}{k^2+3k+2} = \frac{1}{k+1} - \frac{1}{k+2}$$

SO

$$\sum_{k=0}^{\infty} \frac{1}{k^2+3k+2} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+2} \right)$$

NOW WRITE OUT THE PARTIAL SUMS :

$$S_0 = 1 - \frac{1}{2}$$

$$S_1 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\vdots$$

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{1}{n+2}$$

THUS,

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + 3k + 2} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+2}\right) = 1.$$

THAT'S BASICALLY THE END OF THE EASY EXAMPLES. FOR A WHILE NOW WE WILL CONCENTRATE ON JUST DECIDING WHETHER OR NOT A SERIES CONVERGES AND NOT WORRY SO MUCH ABOUT WHAT IT CONVERGES TO, E.G., IT WILL SOON BE QUITE EASY TO SHOW THAT $\sum_{k=1}^{\infty} \frac{1}{k^2}$ CONVERGES, BUT IT TAKES A GREAT DEAL OF WORK TO FIND WHAT IT CONVERGES TO ($\frac{\pi^2}{6}$ ACTUALLY).