

INFINITE SEQUENCES

EVENTUAL GOAL (FAR DOWN THE ROAD) : TO HAVE AN ENTIRELY NEW WAY OF LOOKING AT "FUNCTIONS", E.G., WE KNOW THAT, AT LEAST NEAR $x=0$,

$$e^x \approx 1+x$$

$$e^x \approx 1+x+\frac{1}{2}x^2$$

$$e^x \approx 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$$

$$e^x \approx \sum_{k=0}^n \frac{1}{k!} x^k$$

AND THAT THE APPROXIMATIONS GET BETTER AS YOU GO DOWN THE LIST. WE WOULD LIKE TO MAKE SOME SENSE OF "TAKING THE LIMIT" AND WRITING

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

BUT WE HAVE TO LEARN FIRST ABOUT "SEQUENCES" AND "SERIES".

SEQUENCES :

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

2. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

3. $1, -1, 1, -1, \dots$

4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

5. $3, 3, 3, 3, \dots$

6. $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots$

7. $1, 3, 5, 7, \dots$

OR, IN GENERAL,

$$a_1, a_2, a_3, a_4, \dots$$

IT'S COMMON TO DESCRIBE A SEQUENCE BY JUST GIVING ITS nTH TERM (ALSO CALLED GENERAL TERM):

$$\{a_n\}_{n=1}^{\infty}$$

E.G., FOR THE EXAMPLES ABOVE

1. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ $a_n = \frac{1}{n}, n = 1, 2, 3, 4, \dots$

2. $\frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \dots$ $a_n = \frac{1}{2^{n-1}}, n = 1, 2, 3, 4, \dots$

OR

$$a_n = \frac{1}{2^n}, n = 0, 1, 2, 3, \dots$$

3. $(-1)^0, (-1)^1, (-1)^2, (-1)^3, \dots$ $a_n = (-1)^{n-1}, n = 1, 2, 3, 4, \dots$

4. $\frac{1}{1+1}, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \dots$ $a_n = \frac{n}{n+1}, n=1,2,\dots$

5. $3, 3, 3, 3, \dots$ $a_n = 3, n=1,2,\dots$

6. $(-1)^0 \frac{1}{1+1}, (-1)^1 \frac{2}{2+1}, (-1)^2 \frac{3}{3+1}, (-1)^4 \frac{4}{4+1}, \dots$
 $a_n = (-1)^{n-1} \frac{n}{n+1}, n=1,2,\dots$

7. $2(1)-1, 2(2)-1, 2(3)-1, 2(4)-1, \dots$
 $a_n = 2n-1, n=1,2,\dots$

SOME SEQUENCES "APPROACH" A NUMBER AS YOU MOVE OUT THE SEQUENCE (E.G., $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ "APPROACHES" 0) AND SOME DO NOT (E.G., $1, -1, 1, -1, \dots$ OR $1, 3, 5, 7, \dots$)

A SEQUENCE $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, a_4, \dots\}$ IS SAID TO CONVERGE TO A NUMBER L (CALLED THE LIMIT OF THE SEQUENCE) IF ANY INTERVAL AROUND L (HOWEVER SMALL) CONTAINS ALL THE TERMS OF THE SEQUENCE BEYOND SOME POINT.



IN THIS CASE WE WRITE

$$L = \lim_{n \rightarrow \infty} a_n$$

IF NO SUCH NUMBER EXISTS WE SAY THAT THE SEQUENCE DIVERGES.

EXAMPLES :

1. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

2. $\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} = 0$

3. $1, -1, 1, -1, \dots$ DIVERGES

4. $\lim_{n \rightarrow \infty} \frac{n}{n+1}$

NOTICE THAT THIS LOOKS JUST LIKE A
LIMIT WE COMPUTED A LONG TIME AGO

$$\lim_{x \rightarrow \infty} \frac{x}{x+1}$$

FOR WHICH WE EITHER (1) DIVIDED
TOP AND BOTTOM BY x , OR (2) USED
L'HÔPITAL'S RULE.

EVEN THOUGH A SEQUENCE IS NOT THE
SAME THING AS A FUNCTION OF x
THE BEHAVIOR IS THE SAME AND WE
WILL NOT BOTHER TO CHANGE THE
 n 'S TO x 'S.

9. $\{3^n\}_{n=1}^{\infty} = \{3, 3^2, 3^3, \dots\}$ CLEARLY DIVERGES, BUT

$\{(\frac{1}{2})^n\}_{n=1}^{\infty} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ CONVERGES TO 0

IN GENERAL : $\{x^n\}_{n=1}^{\infty}$ CONVERGES TO 0 IF

$-1 < x < 1$, CONVERGES TO 1 IF $x=1$

AND DIVERGES FOR EVERY OTHER

VALUE OF x .

10. $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$

THIS TIME WE'LL USE THE SQUEEZE THEOREM .

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n} \right) \leq \frac{1}{n}$$

① THIS GOES TO 0.

② AND THIS GOES TO 0.

③ THIS IS TRAPPED BETWEEN 0 AND $\frac{1}{n}$ SO THE SQUEEZE THEOREM SAYS IT MUST GO TO 0 ALSO.