# L'HÔPITAL'S RULE

RECALL : INDETERMINATE FORMS :

$\frac{0}{0}$ $\qquad \lim \frac{f(x) \to 0}{g(x) \to 0}$

$\frac{\infty}{\infty}$ $\qquad \lim \frac{f(x) \to \pm\infty}{g(x) \to \pm\infty}$

"EXTRA WORK REQUIRED" TO DETERMINE THE LIMIT. SOMETIMES THIS IS EASY (E.G., $\lim_{x \to 2} \frac{x^2-4}{x-2}$ ) AND SOMETIMES IT'S NOT (E.G., $\lim_{x \to 0} \frac{\sin x}{x}$ ).

THIS SECTION INTRODUCES A NEW TOOL FOR COMPUTING SUCH LIMITS. TO SEE WHERE IT COMES FROM, THINK ABOUT THE FOLLOWING SITUATION :

$$\lim_{x \to a} \frac{f(x) \to 0}{g(x) \to 0}$$

WHERE $f(x)$ AND $g(x)$ ARE DIFFERENTIABLE (AND THEREFORE CONTINUOUS SO $f(a) = \lim_{x \to a} f(x) = 0$ AND $g(a) = \lim_{x \to a} g(x) = 0$ ).

THEN

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$ (PROVIDED THE DENOMINATOR IS NOT ZERO )

$$= \frac{f'(a)}{g'(a)}$$

$$= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}$$ ( PROVIDED $f'(x)$ AND $g'(x)$ ARE ALSO CONTINUOUS )

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

E.G., $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x^2-4)'}{(x-2)'} = \lim_{x\to 2} \frac{2x}{1} = 2\cdot 2 = 4$

MORE CAREFUL ARGUMENTS SHOW THAT $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ FOR ESSENTIALLY ALL OF THE VARIOUS TYPES OF LIMITS WE HAVE COMPUTED ( $x\to a$, $x \to a^+$, $x\to a^-$, $x\to\infty$, $x\to -\infty$ ) PROVIDED THAT $\lim \frac{f(x)}{g(x)}$ IS OF TYPE $\frac{0}{0}$ OR $\frac{\infty}{\infty}$ . THIS IS CALLED

**L'HÔPITAL'S RULE** : IF EITHER

$$\lim \frac{f(x)}{g(x)} \begin{matrix} \to 0 \\ \to 0 \end{matrix} \qquad \text{OR} \qquad \lim \frac{f(x)}{g(x)} \begin{matrix} \to \pm\infty \\ \to \pm\infty \end{matrix}$$

THEN

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

( INCLUDING THE CASE IN WHICH $\lim \frac{f'(x)}{g'(x)} = \pm\infty$ )

**NOTE** : THE ORIGINAL LIMIT MUST BE INDETERMINATE OF TYPE $\frac{0}{0}$ OR $\frac{\infty}{\infty}$ , E.G., $\lim\limits_{x\to 0} \frac{x+6}{x+2} = 3$ AND THIS IS NOT THE SAME AS $\lim\limits_{x\to 0} \frac{(x+6)'}{(x+2)'} = \lim\limits_{x\to 0} 1 = 1$ .

**EXAMPLES** :

1. $$\lim_{x\to 0} \frac{\sin 2x}{\sin 5x} \begin{matrix} \to 0 \\ \to 0 \end{matrix} = \lim_{x\to 0} \frac{2\cos 2x}{5\cos 5x} = \frac{2\cdot 1}{5\cdot 1} = \frac{2}{5}$$

2. $$\lim_{x\to\infty} \frac{e^{3x}}{x} \begin{matrix} \to \infty \\ \to \infty \end{matrix} = \lim_{x\to\infty} \frac{3e^{3x}}{1} = \infty$$

3. $$\lim_{x\to 0} \frac{e^{x}-1}{x^{3}} \begin{matrix} \to 0 \\ \to 0 \end{matrix} = \lim_{x\to 0} \frac{e^{x}}{3x^{2}} \begin{matrix} \to 1 \\ \to 0 \end{matrix} \qquad \text{DNE}$$

$$e^{x} > 0 \quad \text{AND} \quad 3x^{2} > 0 \implies$$

$$\lim_{x\to 0} \frac{e^{x}-1}{x^{3}} = \infty$$

4. $$\lim_{x\to 0^-} \frac{\tan x}{x^2}\begin{matrix}\to 0\\ \to 0\end{matrix} = \lim_{x\to 0^-}\frac{\sec^2 x}{2x}\begin{matrix}\to 1\\ \to 0\end{matrix}$$ DONE

$2x \to 0$ THROUGH NEGATIVE VALUES SO

$$\lim_{x\to 0^-}\frac{\tan x}{x^2} = -\infty$$

5. $$\lim_{x\to 0^+}\frac{\ln x}{\csc x}\begin{matrix}\to -\infty\\ \to \infty\end{matrix} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x} =$$

$$\lim_{x\to 0^+} -\frac{\sin^2 x}{x\cos x}\begin{matrix}\to 0\\ \to 0\end{matrix} = \lim_{x\to 0^+} -\frac{2\sin x\cos x}{-x\sin x + \cos x}$$

$$= -\frac{2(0)(1)}{0+1} = -\frac{0}{1} = 0$$

## OTHER TYPES OF INDETERMINATE FORMS :

$0\cdot\infty$ $$\lim \underset{\substack{\downarrow\\ 0}}{f(x)}\,\underset{\substack{\downarrow\\ \pm\infty}}{g(x)}$$

$\infty-\infty$ $$\lim \left(\underset{\substack{\downarrow\\ \infty}}{f(x)} - \underset{\substack{\downarrow\\ \infty}}{g(x)}\right)$$

$0^0$ $$\lim \underset{\substack{\downarrow\\ 0}}{f(x)}^{\,g(x)\to 0}$$

$$\infty^0 \qquad \lim \underset{\substack{\downarrow\\ \pm\infty}}{f(x)}^{\,g(x)\to 0}$$

$$1^\infty \qquad \lim \underset{\substack{\downarrow\\ 1}}{f(x)}^{\,g(x)\to \infty}$$

THE FIRST TWO TYPES ARE HANDLED BY ALGEBRAICALLY TURNING THE PRODUCT / DIFFERENCE INTO A QUOTIENT AND USING L'HÔPITAL'S RULE.

EXAMPLES :

6. $$\lim_{x\to 0^+} \underset{\substack{\downarrow\\ 0}}{x}\ \underset{\substack{\downarrow\\ -\infty}}{\ln x}$$

$$= \lim_{x\to 0^+} \frac{\ln x \ \to -\infty}{\frac{1}{x} \ \to \ \infty} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

7. $$\lim_{x\to \frac{\pi}{4}^-} \underset{\substack{\downarrow\\ 0}}{(1-\tan x)}\ \underset{\substack{\downarrow\\ \infty}}{\sec 2x}$$

$$= \lim_{x\to \frac{\pi}{4}^-} \frac{1-\tan x \ \to 0}{\cos 2x \ \to 0} = \lim_{x\to \frac{\pi}{4}^-} \frac{-\sec^2 x}{-2\sin 2x} = \frac{\sec^2 \frac{\pi}{4}}{2\sin\frac{\pi}{2}} = \frac{2}{2\cdot 1} = 1$$

8. $$\lim_{x\to 0^+}\left(\underset{\overset{\downarrow}{\infty}}{\frac{1}{x}} - \underset{\overset{\downarrow}{\infty}}{\frac{1}{\sin x}}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{1}{x}\,\frac{\sin x}{\sin x} - \frac{1}{\sin x}\,\frac{x}{x}\right) = \lim_{x\to 0^+}\frac{\sin x - x \;\to 0}{x\sin x \;\to 0}$$

$$= \lim_{x\to 0^+}\frac{\cos x - 1 \;\longrightarrow 0}{x\cos x + \sin x \;\longrightarrow 0} = \lim_{x\to 0^+}\frac{-\sin x}{-x\sin x + \cos x + \cos x}$$

$$= \frac{-0}{0+1+1} = \frac{0}{2} = 0$$

<u>NOTE</u>: THERE'S NOTHING INDETERMINATE ABOUT

$$\lim_{x\to 0^-}\left(\underset{\overset{\downarrow}{\infty}}{\frac{1}{x^2}} - \underset{\overset{\downarrow}{-\infty}}{\frac{1}{\sin x}}\right) = \infty$$

THE "EXPONENTIAL TYPE" INDETERMINATE FORMS ARE HANDLED BY TAKING LOGARITHMS, EVALUATING THE LIMIT OF THE LOGARITHM, AND THEN EXPONENTIATING TO GET THE ORIGINAL LIMIT.

EXAMPLES:

9. $$\lim_{x\to\infty} \underset{\substack{\downarrow\\ 1}}{\left(1+\frac{1}{x}\right)}^{x\,\to\,\infty}$$

LET $y = \left(1+\frac{1}{x}\right)^x$ (WE WANT TO FIND $\lim_{x\to\infty} y$ )

THEN

$$\ln y = \ln\left(1+\frac{1}{x}\right)^x$$

$$\ln y = x \ln\left(1+\frac{1}{x}\right)$$

$$\ln y = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \begin{matrix}\to 0\\ \to 0\end{matrix}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$$

THUS,

$$\lim_{x\to\infty} \ln y = 1$$

SO

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^1 = e$$

10. $\lim\limits_{x\to 0^+} \underset{\substack{\downarrow \\ 0}}{x}^{x \,\to\, 0}$

LET $y = x^x$

$$\ln y = \ln(x^x)$$

$$\ln y = x \ln x$$

$$\ln y = \frac{\ln x}{\frac{1}{x}}$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\overset{\to\, -\infty}{\ln x}}{\underset{\to\, \infty}{\frac{1}{x}}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

SO

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$$