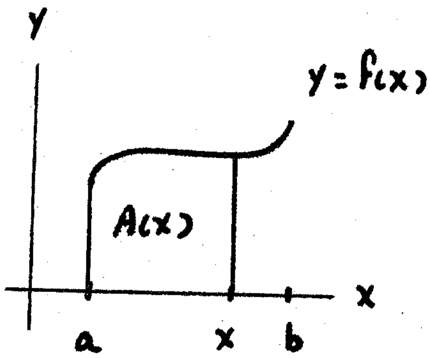
INDEFINITE INTEGRALS

LAST TIME WE DESCRIBED A LONG AND ELABORATE PROCEDURE FOR COMPUTING AREAS (AND NET SIGNED AREAS) FOR A FUNCTION $f(x)$ OVER AN INTERVAL $[a,b]$.

WE WILL RETURN TO THIS PROCEDURE A BIT LATER.

NOW I WANT TO REMIND YOU WHY WE CARE ABOUT SUCH AREAS :



$$A'(x) = f(x)$$

THIS IS THE MOTIVATION BEHIND THE FOLLOWING DEFINITION :

GIVEN A FUNCTION $f(x)$ ON SOME INTERVAL I

AN <u>ANTIDERIVATIVE</u> FOR $f(x)$ ON I IS ANOTHER

FUNCTION $F(x)$ SATISFYING

$$F'(x) = f(x)$$

FOR ALL x IN I.

E.G., IF $f(x) = 3x^2$, THEN ALL OF THE FOLLOWING ARE ANTIDERIVATIVES FOR $f(x)$ ON $(-\infty, \infty)$:

$$x^3, \quad x^3 + 5, \quad x^3 - \frac{\pi^2}{6}, \ldots$$

$$x^3 + C \qquad \text{FOR ANY CONSTANT } C$$

<u>NOTE</u>: LAST TERM WE PROVED, FROM THE MEAN VALUE THEOREM, THAT $x^3 + C$ DESCRIBES <u>ALL</u> OF THE ANTIDERIVATIVES OF $3x^2$ ON $(-\infty, \infty)$.

$$f(x) \xrightarrow{\text{DIFFERENTIATION}} f'(x)$$

$$f(x) \xrightarrow[\text{INTEGRATION}]{\text{ANTIDIFFERENTIATION}} F(x) \text{ FOR WHICH } F'(x) = f(x)$$

INTEGRAL NOTATION:

$$\int f(x)\,dx = \text{COLLECTION OF ALL ANTIDERIVATIVES FOR } f(x) \text{ ON } I$$

$$= F(x) + C$$

WHERE $F(x)$ IS ANY FUNCTION SATISFYING $F'(x) = f(x)$ ON I AND C IS AN ARBITRARY CONSTANT

E.G.,

$$\int 3x^2\,dx = x^3 + C$$

THE SYMBOLS :

$$\underbrace{\int}_{\text{INTEGRAL SIGN}} \underbrace{f(x)}_{\text{INTEGRAND}} \overbrace{dx}^{\text{DIFFERENTIAL}}$$

$\int f(x)\,dx$ = <u>INDEFINITE</u> <u>INTEGRAL</u> OF $f(x)$

<u>A</u> <u>FEW</u> <u>EXAMPLES</u> :

$$\int \cos x\,dx = \sin x + C \qquad \left(\frac{d}{dx}(\sin x) = \cos x\right)$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \qquad \left(\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}\right)$$

$$\int x^5\,dx = \frac{1}{6}x^6 + C \qquad \left(\frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5\right)$$

$$\int t^5\,dt = \frac{1}{6}t^6 + C \qquad \left(\frac{d}{dt}\left(\frac{1}{6}t^6\right) = t^5\right)$$

BASIC TABLE OF INTEGRALS

1. $\int \cos x\,dx = \sin x + C$

2. $\int \sin x\,dx = -\cos x + C$

3. $\int \sec^2 x\,dx = \tan x + C$

4. $\int \csc^2 x\,dx = -\cot x + C$

5. $\int \sec x \tan x\,dx = \sec x + C$

6. $\int \csc x \cot x\,dx = -\csc x + C$

7. $\int e^x\,dx = e^x + C$

8. $\int b^x\,dx = \frac{b^x}{\ln b} + C$

9. $\int 1\,dx = \int dx = x + C$

10. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

    PROVIDED $n \neq -1$

11. $\int \frac{1}{x}\,dx = \ln|x| + C$

12. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

13. $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

14. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}|x| + C$

AND A FEW SIMPLE RESULTS ON PUTTING THEM TOGETHER: SINCE $(F(x) \pm G(x))' = F(x) \pm G(x)$ AND $(c\,F(x))' = c\,F'(x)$,

15. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

16. $\int c\,f(x)\,dx = c\int f(x)\,dx$

EXAMPLES :

1. $\int x^5 dx = \frac{x^{5+1}}{5+1} + C = \frac{1}{6}x^6 + C$

2. $\int \sqrt{t}\, dt = \int t^{\frac{1}{2}} dt = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3} t^{\frac{3}{2}} + C$

3. $\int (x^2 + x)\, dx = \int x^2 dx + \int x\, dx$

$= \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} + C$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$

NOTE: ONE ARBITRARY CONSTANT WILL SUFFICE SINCE THE SUM OF TWO IS JUST ANOTHER ARBITRARY CONSTANT

4. $\int \left(\frac{1}{\sqrt{u}} - 4\sec^2 u\right) du = \int (u^{-\frac{1}{2}} - 4\sec^2 u)\, du =$

$\int u^{-\frac{1}{2}} du - 4\int \sec^2 u\, du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 4\tan u + C =$

$2u^{\frac{1}{2}} - 4\tan u + C = 2\sqrt{u} - 4\tan u + C$

5. $\int \frac{5}{\sqrt{1-t^2}}\, dt = 5\int \frac{1}{\sqrt{1-t^2}}\, dt = 5 \arcsin t + C$

6. $\int \frac{x^3+3x-1}{x^2}\,dx = \int \left(\frac{x^3}{x^2} + \frac{3x}{x^2} - \frac{1}{x^2}\right)dx =$

$\int (x + 3(\frac{1}{x}) - x^{-2})\,dx = \int x\,dx + 3\int \frac{1}{x}\,dx - \int x^{-2}\,dx =$

$\frac{x^{1+1}}{1+1} + 3\ln|x| - \frac{x^{-2+1}}{-2+1} + C = \frac{1}{2}x^2 + 3\ln|x| - \frac{x^{-1}}{-1} + C =$

$\frac{1}{2}x^2 + 3\ln|x| + \frac{1}{x} + C$

7. SOLVE THE <u>INITIAL</u> <u>VALUE</u> <u>PROBLEM</u> $\begin{cases} \frac{dy}{dx} = \sec^2 x - \sin x \\ y(\frac{\pi}{4}) = 1 \end{cases}$

ALL THIS ASKS FOR IS AN ANTIDERIVATIVE y FOR $\sec^2 x - \sin x$ WHOSE VALUE AT $\frac{\pi}{4}$ IS 1.

ALL ANTIDERIVATIVES FOR $\sec^2 x - \sin x$ ARE

$$\int (\sec^2 x - \sin x)\,dx = \int \sec^2 x\,dx - \int \sin x\,dx =$$

$$\tan x - (-\cos x) + C = \tan x + \cos x + C$$

THUS, FOR SOME CHOICE OF C,

$$y(x) = \tan x + \cos x + C$$

AT $x = \frac{\pi}{4}$,

$$1 = y(\tfrac{\pi}{4}) = \tan\tfrac{\pi}{4} + \cos\tfrac{\pi}{4} + C = 1 + \tfrac{\sqrt{2}}{2} + C$$

SO

$$C = -\frac{\sqrt{2}}{2}$$

AND

$$y(x) = \tan x + \cos x - \frac{\sqrt{2}}{2}.$$