IMPLICIT DIFFERENTIATION

MOTIVATION : COMPARE

1.	$x^2 - y^3 = 3$	WITH	$y = \sqrt[3]{x^2-3}$
2.	$x^2 + y^2 = 1$	WITH	$y = \pm\sqrt{1-x^2}$
3.	$x^3 + y^3 = 3xy$	WITH	?

y IS GIVEN IMPLICITLY AS A FUNCTION OF x | y IS GIVEN EXPLICITLY AS A FUNCTION OF x

NOTE THAT IT'S EASY TO COMPUTE y' DIRECTLY FROM $y = \sqrt[3]{x^2-3}$:

$$y' = \left((x^2-3)^{\frac{1}{3}}\right)' = \frac{1}{3}(x^2-3)^{-\frac{2}{3}}(2x)$$

$$= \frac{2x}{3(x^2-3)^{2/3}}$$

BUT ALSO NOTICE THAT YOU CAN GET y' DIRECTLY FROM $x^2 - y^3 = 3$ BY REGARDING y AS SOME "UNKNOWN" FUNCTION OF x, DIFFERENTIATING BOTH SIDES AND USING THE CHAIN RULE :

$$(x^2 - y^3)' = (3)'$$

$$(x^2)' - (y^3)' = 0$$

$$2x - 3y^2y' = 0 \qquad \text{(CHAIN RULE !)}$$

$$y' = \frac{2x}{3y^2}$$

THE "y" APPEARS HERE BECAUSE WE HAVEN'T SOLVED $x^2 - y^3 = 3$ FOR y.

IF WE DID SOLVE FOR y WE'D GET $y = (x^2-3)^{\frac{1}{3}}$ AND, SUBSTITUTING THIS, THE TWO ANSWERS FOR y' WOULD BE THE SAME.

SOMETIMES HAVING THE y IN THE EXPRESSION FOR y' IS A GOOD THING:

<u>EXAMPLE</u>: FIND THE SLOPE OF THE TANGENT LINE TO THE CIRCLE $x^2 + y^2 = 1$ AT AN ARBITRARY POINT (a, b) (WITH $b \neq 0$)

> <u>NOTE</u>: DON'T KNOW WHETHER (a,b) IS ON THE TOP HALF ($y = \sqrt{1-x^2}$) OR THE BOTTOM HALF ($y = -\sqrt{1-x^2}$).

FIND y' AS WE DID FOR $x^2 - y^3 = 3$:

$$(x^2 + y^2)' = 1'$$

$$(x^2)' + (y^2)' = 0$$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y} \qquad (y \neq 0)$$

THUS, THE SLOPE AT (a,b) IS

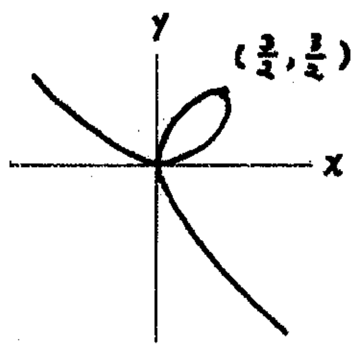
$$y'(a,b) = -\frac{a}{b}.$$

THIS PROCEDURE FOR FINDING y' IS EVEN MORE USEFUL WHEN IT'S SIMPLY IMPOSSIBLE TO SOLVE FOR y EXPLICITLY, E.G., FOR

$$x^3 + y^3 = 3xy$$
$$(x^3 + y^3)' = (3xy)'$$
$$(x^3)' + (y^3)' = 3(xy)'$$
$$3x^2 + 3y^2y' = 3(xy' + y(x)')$$
$$x^2 + y^2y' = xy' + y$$
$$y^2y' - xy' = y - x^2$$
$$(y^2 - x)y' = y - x^2$$
$$y' = \frac{y - x^2}{y^2 - x}$$

<u>MORE</u> <u>EXAMPLES</u> :

1. THE GRAPH OF $x^3 + y^3 = 3xy$ IS CALLED THE <u>FOLIUM</u> <u>OF</u> <u>DESCARTES</u> AND LOOKS LIKE THIS :



(A "FOLIUM" IS A LEAF AND DESCARTES IS SOME FRENCH GUY)

THE SLOPE OF THE TANGENT LINE AT $(\frac{3}{2}, \frac{3}{2})$ IS

$$y'(\tfrac{3}{2}, \tfrac{3}{2}) = \frac{\frac{3}{2} - (\frac{3}{2})^2}{(\frac{3}{2})^2 - \frac{3}{2}} = -1$$

2. FIND y' IF y IS GIVEN IMPLICITLY BY $x^2 = \frac{\cot y}{1+\csc y}$

$$(x^2)' = \left(\frac{\cot y}{1+\csc y}\right)'$$

$$2x = \frac{(1+\csc y)((-\csc^2 y)y') - (\cot y)((-\csc y \cot y)y')}{(1+\csc y)^2}$$

$$2x = \frac{(-\csc^2 y - \csc^3 y + \csc y \cot^2 y)y'}{(1+\csc y)^2}$$

$$2x = \frac{-\csc y\,(\csc y + \csc^2 y - \cot^2 y)y'}{(1+\csc y)^2}$$

$$2x = \frac{-\csc y\,(\csc y + 1)\,y'}{(1+\csc y)^2}$$

REMEMBER: $\csc^2 y - \cot^2 y = 1$

$$2x = \frac{(-\csc y)y'}{1+\csc y}$$

SO $$y' = -\frac{2x(1+\csc y)}{\csc y} = -2x(\sin y + 1)$$

3. FIND y'' IF $x^3 + y^3 = 1$

$$(x^3+y^3)' = (1)'$$

$$3x^2 + 3y^2 y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

SO $$y'' = (y')' = \left(-\frac{x^2}{y^2}\right)' = -\frac{y^2(2x) - x^2(2yy')}{(y^2)^2}$$

$$= -\frac{2xy^2 - 2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4} = -\frac{2xy^2 + 2x^4\left(\frac{1}{y}\right)}{y^4}$$

$$= -\frac{2xy^3 + 2x^4}{y^5}$$