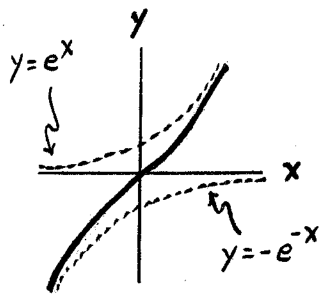
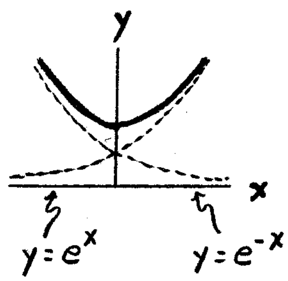
THE HYPERBOLIC FUNCTIONS

HYPERBOLIC SINE	:	$\sinh x$	$=$	$\frac{1}{2}(e^{x} - e^{-x})$
HYPERBOLIC COSINE	:	$\cosh x$	$=$	$\frac{1}{2}(e^{x} + e^{-x})$
HYPERBOLIC TANGENT	:	$\tanh x$	$=$	$\frac{\sinh x}{\cosh x}$
HYPERBOLIC COTANGENT	:	$\coth x$	$=$	$\frac{\cosh x}{\sinh x}$
HYPERBOLIC SECANT	:	$\operatorname{sech} x$	$=$	$\frac{1}{\cosh x}$
HYPERBOLIC COSECANT	:	$\operatorname{csch} x$	$=$	$\frac{1}{\sinh x}$

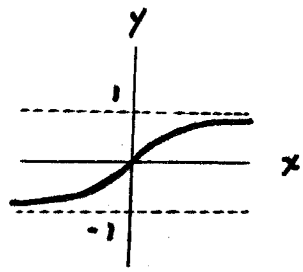


$y = \sinh x$



$y = \cosh x$

(CATENARY)



$y = \tanh x$

HYPERBOLIC FUNCTIONS SATISFY MANY IDENTITIES ANALOGOUS TO STANDARD TRIGONOMETRIC IDENTITIES, E.G.,

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

ETC.

SAMPLE PROOF :

$$\cosh^2 x - \sinh^2 x = \left(\tfrac{1}{2}(e^x + e^{-x})\right)^2 - \left(\tfrac{1}{2}(e^x - e^{-x})\right)^2$$

$$= \tfrac{1}{4}\left((e^x)^2 + 2e^x e^{-x} + (e^{-x})^2\right)$$

$$- \tfrac{1}{4}\left((e^x)^2 - 2e^x e^{-x} + (e^{-x})^2\right)$$

$$= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x}) - \tfrac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x})$$

$$= 1$$

DERIVATIVES :

$$(\sinh x)' = \left(\tfrac{1}{2}(e^x - e^{-x})\right)' = \tfrac{1}{2}\left((e^x)' - (e^{-x})'\right)$$

$$= \tfrac{1}{2}(e^x - (-e^{-x})) = \tfrac{1}{2}(e^x + e^{-x})$$

$$= \cosh x$$

SIMILAR CALCULATIONS GIVE THE FOLLOWING :

$$(\sinh x)' = \cosh x$$
$$(\cosh x)' = \sinh x$$
$$(\tanh x)' = \operatorname{sech}^2 x$$
$$(\coth x)' = -\operatorname{csch}^2 x$$
$$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$$
$$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$$

NOTICE THE TWO UNEXPECTED SIGNS

NOW COMBINE THESE WITH THE CHAIN RULE.

EXAMPLES :

1. $\left(\cosh(x^3)\right)' = \sinh(x^3)\,(x^3)' = 3x^2\sinh(x^3)$

2. $\left(\ln(\tanh x)\right)' = \dfrac{1}{\tanh x}\,(\tanh x)' = \coth x\,\operatorname{sech}^2 x$

INTEGRALS :

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x\,dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$$

NOTICE THE TWO UNEXPECTED SIGNS

EXAMPLES :

1. $\displaystyle\int \sinh^5 x \cosh x\,dx = \int u^5\,du = \frac{1}{6}u^4 + C$

$u = \sinh x$

$du = \cosh x\,dx$

$\displaystyle = \frac{1}{6}\sinh^4 x + C$

2. $\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{\cosh x} \sinh x \, dx$

$u = \cosh x$
$du = \sinh x \, dx$

$$= \int \frac{1}{u} du = \ln |u| + C$$

$$= \ln |\cosh x| + C$$

$$= \ln (\cosh x) + C$$

3. FIND THE LENGTH OF THE CATENARY $y = \cosh x$ FROM $x = -1$ TO $x = 1$.

$$L = \int_{-1}^{1} \sqrt{1 + ((\cosh x)')^2} \, dx = \int_{-1}^{1} \sqrt{1 + \sinh^2 x} \, dx$$

$$= \int_{-1}^{1} \sqrt{\cosh^2 x} \, dx = \int_{-1}^{1} \cosh x \, dx = \sinh x \Big|_{-1}^{1}$$

$$= \sinh 1 - \sinh(-1)$$

$$= \tfrac{1}{2}(e^1 - e^{-1}) - \tfrac{1}{2}(e^{-1} - e^1) = \tfrac{1}{2}(e - \tfrac{1}{e} - \tfrac{1}{e} + e)$$

$$= e - \frac{1}{e}$$

INVERSES FOR $y = \sinh x$, $y = \cosh x$ (ON $x \geq 0$), AND $y = \tanh x$:

FOR EXAMPLE,

$$y = \sinh x$$
$$y = \tfrac{1}{2}(e^x - e^{-x})$$
$$2y = e^x - e^{-x}$$
$$(2y)e^x = (e^x)^2 - 1$$
$$(e^x)^2 - (2y)e^x - 1 = 0$$

COMPARE : $z^2 - (2y)z - 1 = 0$ (QUADRATIC EQUATION FOR z)

SOLVE FOR e^x WITH QUADRATIC FORMULA :

$$e^x = \frac{-(-2y) \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{2y \pm \sqrt{4y^2+4}}{2} = \frac{2y \pm 2\sqrt{y^2+1}}{2}$$
$$= y \pm \sqrt{y^2+1}$$

SINCE $e^x > 0$ WE MUST CHOOSE THE + SIGN.

$$e^x = y + \sqrt{y^2+1}$$
$$x = \ln(y + \sqrt{y^2+1})$$

THUS,

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^2+1}), \quad -\infty < x < \infty$$

SIMILAR CALCULATIONS YIELD THE REST :

$$\sinh^{-1} x = \ln(x+\sqrt{x^2+1}) \quad , \quad -\infty < x < \infty$$

$$\cosh^{-1} x = \ln(x+\sqrt{x^2-1}) \quad , \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad , \quad -1 < x < 1$$

<u>DERIVATIVES OF THE INVERSE FUNCTIONS</u> :

$$\begin{aligned}
(\sinh^{-1} x)' &= (\ln(x+\sqrt{x^2+1}))' \\
&= \frac{1}{x+\sqrt{x^2+1}}(x+(x^2+1)^{\frac{1}{2}})' \\
&= \frac{1}{x+\sqrt{x^2+1}}(1+\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)) \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}+\frac{x}{\sqrt{x^2+1}}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

SIMILARLY ,

$$(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$$

$$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}$$

$$(\tanh^{-1} x)' = \frac{1}{1-x^2}$$

NOW COMBINE THESE WITH THE CHAIN RULE.

EXAMPLES :

1. $$\left(\cosh^{-1}\left(\frac{2}{x}\right)\right)' = \frac{1}{\sqrt{\left(\frac{2}{x}\right)^2-1}}\left(\frac{2}{x}\right)' = \frac{1}{\sqrt{\frac{4}{x^2}-1}}\left(-\frac{2}{x^2}\right)$$

$$= \frac{1}{\sqrt{\frac{4-x^2}{x^2}}}\left(-\frac{2}{x^2}\right) = \frac{1}{\frac{\sqrt{4-x^2}}{x}}\left(-\frac{2}{x^2}\right)$$

$$= \frac{x}{\sqrt{4-x^2}}\left(-\frac{2}{x^2}\right) = -\frac{2}{x\sqrt{4-x^2}}$$

2. $$y = (1 + x\tanh^{-1}x)^{10}$$

$$y' = 10(1+x\tanh^{-1}x)^9(1+x\tanh^{-1}x)'$$

$$= 10(1+x\tanh^{-1}x)^9(0 + x(\tanh^{-1}x)' + \tanh^{-1}x)$$

$$= 10(1+x\tanh^{-1}x)^9\left(\frac{x}{1-x^2} + \tanh^{-1}x\right)$$

INTEGRALS :

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1}x + C = \ln(x+\sqrt{x^2+1}) + C$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}x + C = \ln(x+\sqrt{x^2-1}) + C$$

EXAMPLE :

$$\int \frac{dx}{\sqrt{4x^2-9}} = \int \frac{1}{\sqrt{9\left(\frac{4x^2}{9}-1\right)}}\,dx$$

$$= \int \frac{1}{3\sqrt{\left(\frac{2x}{3}\right)^2-1}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{2x}{3}\right)^2-1}}\,dx$$

$$u = \frac{2x}{3}$$

$$du = \frac{2}{3}\,dx$$

$$= \frac{1}{3}\,\frac{3}{2}\int \frac{1}{\sqrt{u^2-1}}\,du$$

$$= \frac{1}{2}\cosh^{-1}u + C = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{2x}{3} + \sqrt{\left(\frac{2x}{3}\right)^2-1}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{2x}{3} + \sqrt{\frac{4x^2-9}{9}}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{2x+\sqrt{4x^2-9}}{3}\right) + C$$

$$= \frac{1}{2}\ln\left(2x+\sqrt{4x^2-9}\right) - \frac{1}{2}\ln 3 + C$$

$$= \frac{1}{2}\ln\left(2x+\sqrt{4x^2-9}\right) + C$$

(ABSORB $-\frac{1}{2}\ln 3$ INTO THE ARBITRARY CONSTANT)