GEOMETRICAL AND NUMERICAL METHODS

UNFORTUNATE FACT OF LIFE : THE DIFFERENTIAL EQUATION

$$y' = f(x, y)$$

IS GENERALLY IMPOSSIBLE TO SOLVE EXACTLY.

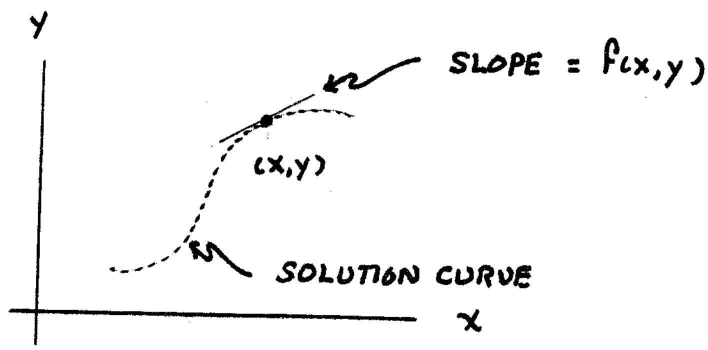
EVEN WHEN THIS IS THE CASE IT IS STILL POSSIBLE TO OBTAIN INFORMATION ABOUT THE SOLUTIONS, E.G.,

- GENERAL, QUALITATIVE BEHAVIOR (SHAPES OF THE INTEGRAL CURVES)
- APPROXIMATE VALUES OF SOLUTIONS

WE WILL CONSIDER ONE SIMPLE METHOD FOR EACH OF THESE.

- DIRECTION FIELDS
- EULER'S METHOD

IDEA BEHIND DIRECTION FIELDS : $y' = f(x, y)$



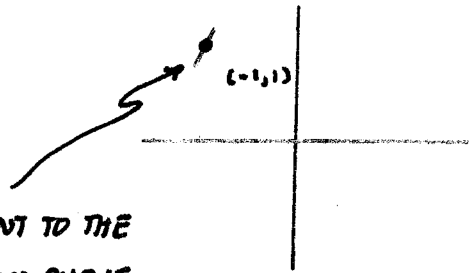
E.G., ANY SOLUTION TO

$$y' = x^2 + y^2$$

THROUGH THE POINT $(-1, 1)$ MUST HAVE SLOPE

$$(-1)^2 + (1)^2 = 2$$

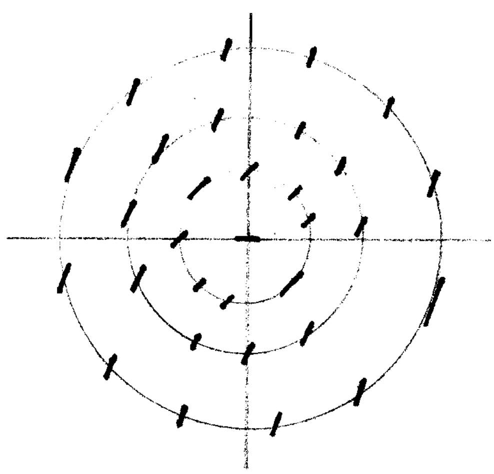
THERE.



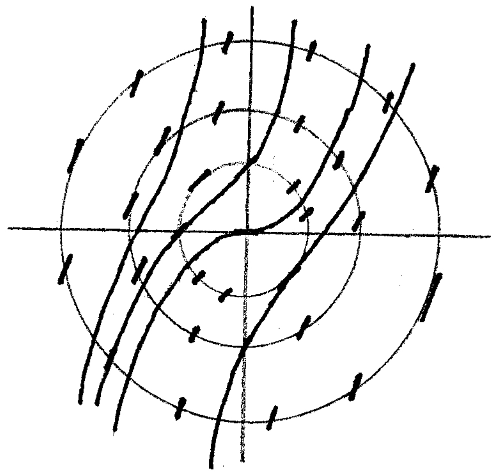
TANGENT TO THE
SOLUTION CURVE
TO $y' = x^2 + y^2$
THROUGH $(-1, 1)$
MUST HAVE SLOPE 2.

SUPPOSE WE DREW (OR HAD A COMPUTER DRAW) LOTS AND LOTS OF THESE LITTLE TANGENT LINES TO POINTS IN THE PLANE INDICATING THE SLOPE OF ANY SOLUTION CURVE AT THESE POINTS.

E.G., NOTING THAT $x^2 + y^2$ IS CONSTANT ON CIRCLES ABOUT THE ORIGIN,



THE PATTERNS OF THESE SLOPES (THE <u>DIRECTION</u> <u>FIELD</u>), SUGGESTS THE SHAPE OF THE CURVES THAT HAVE THESE TANGENTS, I.E., THE INTEGRAL CURVES OF THE EQUATION.



<u>ANOTHER</u> <u>EXAMPLE</u> : $y' = 4y(1-y)$

<u>NOTE</u> : THIS IS AN EXAMPLE OF A <u>LOGISTIC</u> <u>EQUATION</u> AND IS USED TO DESCRIBE A CERTAIN <u>POPULATION</u> <u>MODEL</u> .

$$y = y(t) = \text{POPULATION AT TIME } t$$

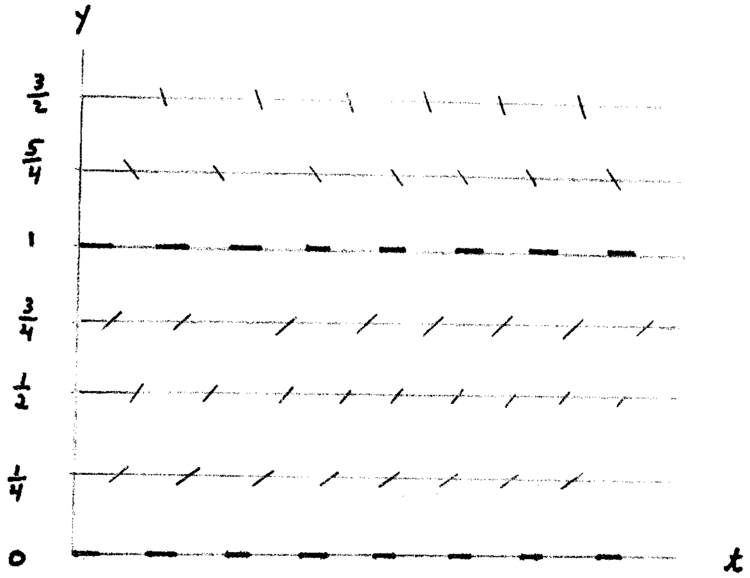
WE ARE INTERESTED IN SOLUTIONS THAT SATISFY AN INITIAL CONDITION

$$y(0) = y_0$$

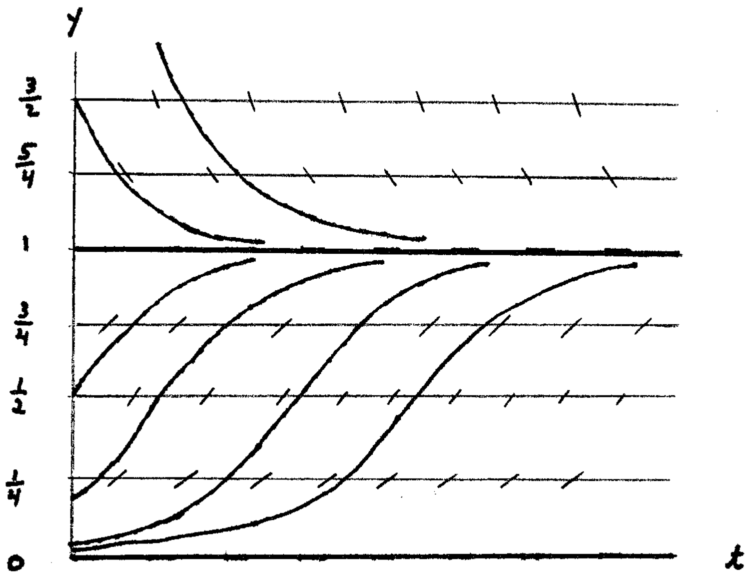
WHERE $y_0 \geq 0$.

NOTE THAT THE RIGHT-HAND SIDE (SLOPE) IS CONSTANT ON HORIZONTAL LINES $y = k$. WE'LL CHOOSE A NUMBER OF THESE FOR DRAWING THE DIRECTION FIELD.

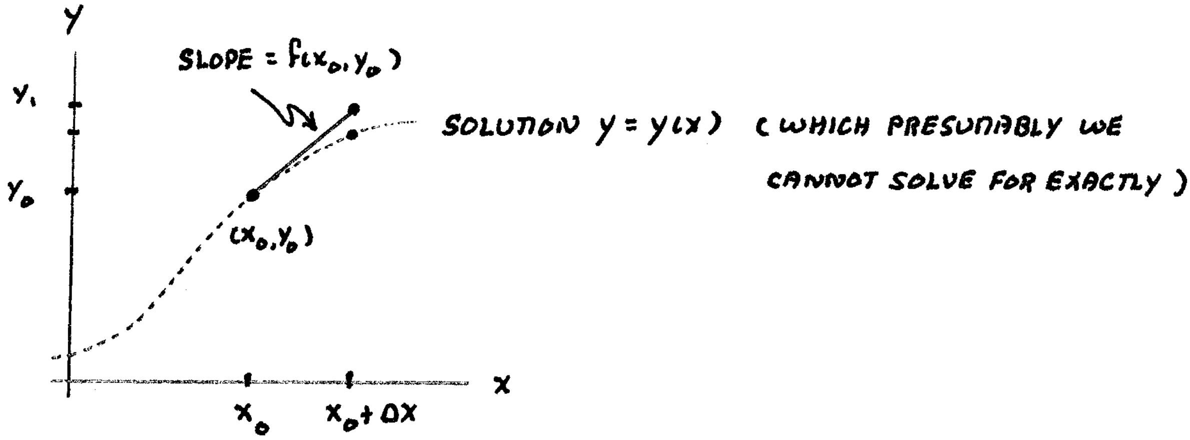
$y = k$	$y' = 4k(1-k)$
0	0
$\frac{1}{4}$	$\frac{3}{4}$
$\frac{1}{2}$	1
$\frac{3}{4}$	$\frac{3}{4}$
1	0
$\frac{5}{4}$	$-\frac{5}{4}$
$\frac{3}{2}$	-3



STABLE POPULATION →



IDEA BEHIND EULER'S METHOD : $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$



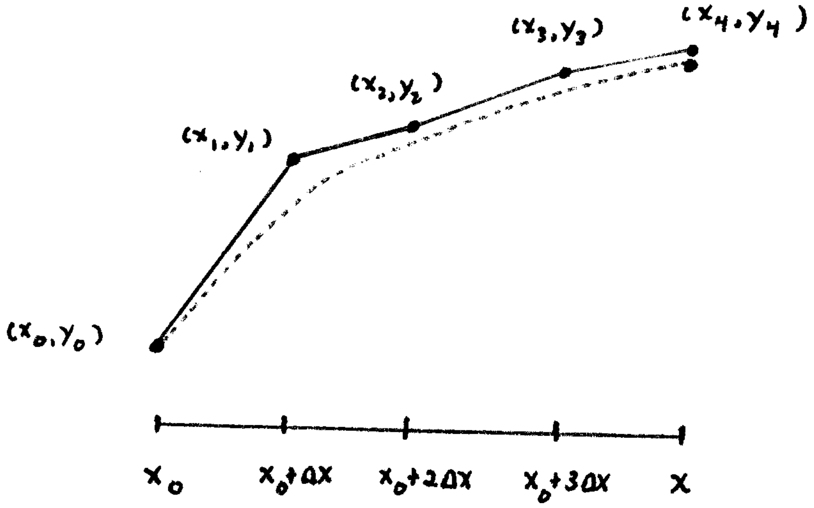
SUPPOSE YOU NEED $y(x_0 + \Delta x)$. IF Δx IS SMALL ENOUGH APPROXIMATE $y(x_0 + \Delta x)$ WITH THE HEIGHT OF THE TANGENT LINE ABOVE $x_0 + \Delta x$:

$$y - y_0 = f(x_0, y_0)(x - x_0)$$

$$x = x_0 + \Delta x$$

$$y_1 = y_0 + f(x_0, y_0)\Delta x \approx y(x_0 + \Delta x)$$

OR, IF YOU NEED $y(x)$ FOR SOME x THAT IS NOT CLOSE TO x_0, APPROACH IT IN "STEPS", E.G.,



SUBDIVIDE $[x_0, X]$ INTO n EQUAL SUBINTERVALS OF LENGTH

$$\Delta x = \frac{X - x_0}{n} \qquad \text{(\underline{STEP} \underline{SIZE})}$$

($n = 4$ IS SHOWN ABOVE)

$$y_1 = y_0 + f(x_0, y_0)\Delta x$$

$$y_2 = y_1 + f(x_1, y_1)\Delta x$$

$$y_3 = y_2 + f(x_2, y_2)\Delta x$$

$$y_4 = y_3 + f(x_3, y_3)\Delta x$$

$$\vdots$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\Delta x \approx y(x)$$

<u>EXAMPLE</u> :

$$\begin{cases} y' = 4y(1-y) \\ y(0) = \frac{1}{4} \end{cases}$$

$$f(x,y) = 4y(1-y)$$

$$(x_0, y_0) = (0, \tfrac{1}{4})$$

APPROXIMATE $y(0.3)$ WITH A STEP SIZE OF $\Delta x = 0.1$ (SO $n = 3$)

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)\Delta x \\ &= \tfrac{1}{4} + f(0, \tfrac{1}{4})(0.1) \\ &= \tfrac{1}{4} + (4(\tfrac{1}{4})(1-\tfrac{1}{4}))(0.1) \end{aligned}$$

$$= \frac{1}{4} + \left(\frac{3}{4}\right)(0.1)$$

$$= 0.25 + 0.075$$

$$= 0.325$$

$$(x_1, y_1) = (0.1, 0.325)$$

$$y_2 = y_1 + f(x_1, y_1)\Delta x$$

$$= 0.325 + \left(\frac{1}{4}(0.325)(1-0.325)\right)(0.1)$$

$$= 0.325 + (0.8775)(0.1)$$

$$= 0.4128$$

$$(x_2, y_2) = (0.2, 0.4128)$$

$$y_3 = y_2 + f(x_2, y_2)\Delta x$$

$$= 0.4128 + \left(\frac{1}{4}(0.4128)(1-0.4128)\right)(0.1)$$

$$= 0.5198$$

SO

$$y(0.3) \approx y_3 = 0.5198$$

<u>NOTE</u>: IN THE NEXT SECTION WE WILL SOLVE THIS EQUATION EXACTLY AND FIND THAT

$$y(0.3) = \frac{0.25\, e^{4(0.3)}}{1 + 0.25(e^{4(0.3)} - 1)} \approx 0.5253$$

(BUT ARRIVING AT 0.5253 ALSO INVOLVES APPROXIMATIONS)