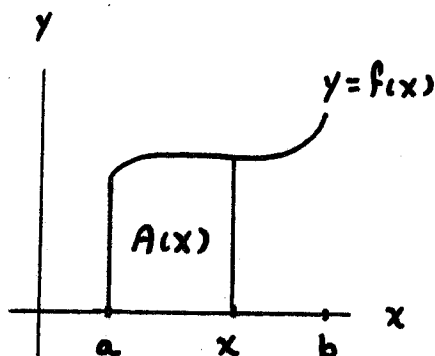


THE FUNDAMENTAL THEOREM OF CALCULUS :

THERE ARE TWO PARTS TO THIS. AS MOTIVATION FOR THE FIRST, RECALL THAT



$$A'(x) = f(x)$$

$$\frac{d}{dx} A(x) = f(x)$$

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

THE FUNDAMENTAL THEOREM OF CALCULUS SAYS THAT THIS IS ALWAYS TRUE :

IF $f(x)$ IS CONTINUOUS ON THE INTERVAL I
AND a IS IN I , THEN

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

I.E., $F(x) = \int_a^x f(t) dt$ IS AN

ANTIDERIVATIVE FOR $f(x)$.

EXAMPLES :

$$1. \frac{d}{dx} \int_1^x \cos t dt = \cos x$$

$$2. \frac{d}{dx} \int_5^x \cos t dt = \cos x$$

3. DEFINE A FUNCTION $y = f(x)$ BY

$$f(x) = \int_{\sqrt{3}}^x \text{ARCTAN } t dt .$$

FIND $f(\sqrt{3})$, $f'(\sqrt{3})$ AND $f''(\sqrt{3})$.

$$f(\sqrt{3}) = \int_{\sqrt{3}}^{\sqrt{3}} \text{ARCTAN } t dt = 0$$

$$\begin{aligned} f'(x) &= \frac{d}{dx} f(x) = \frac{d}{dx} \int_{\sqrt{3}}^x \text{ARCTAN } t dt \\ &= \text{ARCTAN } x \end{aligned}$$

$$f'(\sqrt{3}) = \text{ARCTAN } \sqrt{3} = \frac{\pi}{3}$$

$$f''(x) = (\text{ARCTAN } x)' = \frac{1}{1+x^2}$$

$$f''(\sqrt{3}) = \frac{1}{1+(\sqrt{3})^2} = \frac{1}{4}$$

$$\begin{aligned} 4. \frac{d}{dx} \int_x^1 \frac{\sin t}{t} dt &= \frac{d}{dx} \left(- \int_1^x \frac{\sin t}{t} dt \right) \\ &= - \frac{d}{dx} \int_1^x \frac{\sin t}{t} dt = - \frac{\sin x}{x} \end{aligned}$$

THE OTHER HALF OF THE FUNDAMENTAL THEOREM OF CALCULUS IS VERY CLOSELY RELATED TO THIS.

SUPPOSE $f(x)$ IS CONTINUOUS ON $[a, b]$. THEN $\int_a^x f(t) dt$ IS AN ANTIDERIVATIVE FOR $f(x)$ ON $[a, b]$.

ANY OTHER ANTIDERIVATIVE $F(x)$ FOR $f(x)$ ON $[a, b]$ DIFFERS FROM THIS ONE BY SOME CONSTANT :

$$F(x) = \int_a^x f(t) dt + C$$

NOW NOTICE THAT

$$\begin{aligned} F(b) - F(a) &= \left[\int_a^b f(t) dt + C \right] - \left[\int_a^a f(t) dt + C \right] \\ &= \int_a^b f(t) dt + C - 0 - C \\ &= \int_a^b f(x) dx \end{aligned}$$

IF $f(x)$ IS CONTINUOUS ON $[a, b]$
AND $F(x)$ IS ANY ANTIDERIVATIVE
FOR $f(x)$ ON $[a, b]$, THEN

$$\begin{aligned} \int_a^b f(x) dx &= F(b) - F(a) \\ &= F(x) \Big|_a^b \end{aligned}$$

EXAMPLES :

$$1. \int_1^2 x^3 dx = \frac{1}{4} x^4 \Big|_1^2 = \frac{1}{4} (2^4 - 1^4) = \frac{1}{4} (16 - 1) = \frac{15}{4}$$

$$2. \int_0^{2\pi} \cos x dx = \sin x \Big|_0^{2\pi} = \sin 2\pi - \sin 0 = 0 - 0 = 0$$

$$3. \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \arcsin \frac{1}{2} - \arcsin \left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$4. \int_0^{\frac{\pi}{4}} \tan x dx = \ln |\sec x| \Big|_0^{\frac{\pi}{4}} = \ln |\sec \frac{\pi}{4}| - \ln |\sec 0|$$

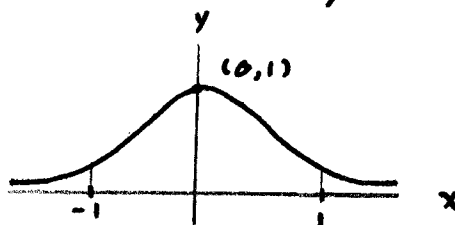
$$= \ln |\sqrt{2}| - \ln |1| = \ln \sqrt{2} = \frac{1}{2} \ln 2.$$

$$5. \int_{-e}^{-1} \frac{1}{x} dx = \ln |x| \Big|_{-e}^{-1} = \ln |-1| - \ln |-e| = \ln 1 - \ln e$$

$$= 0 - 1$$

$$= -1$$

6. FIND THE AREA UNDER THE GRAPH OF $y = f(x) = \frac{1}{1+x^2}$ FROM $x = -1$ TO $x = 1$.



$$A = \int_{-1}^1 \frac{1}{1+x^2} dx = \arctan x \Big|_{-1}^1 = \arctan(1) - \arctan(-1)$$

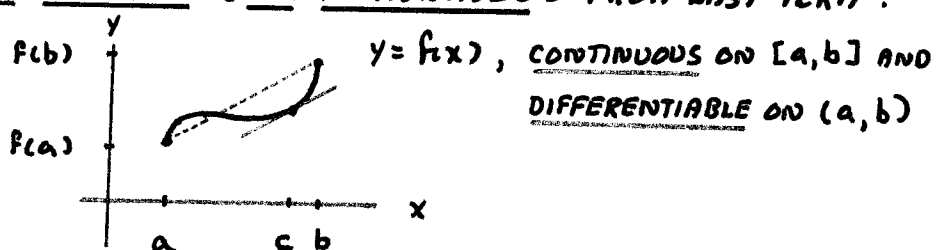
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

NOTICE THAT THE FOLLOWING IS OBVIOUSLY STUPID :

$$\int_{-1}^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{-1}^1 = -\frac{1}{1} - \left(-\frac{1}{-1}\right) \\ = -1 - 1 = -2$$

WHY IS IT OBVIOUSLY STUPID AND WHAT WENT WRONG ?

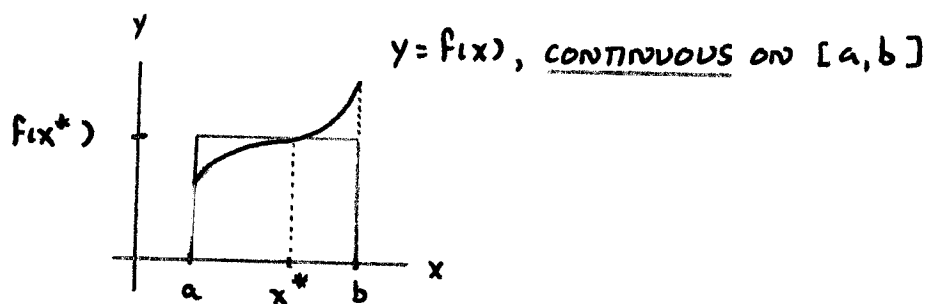
RECALL THE MEAN VALUE THEOREM (FOR DERIVATIVES) FROM LAST TERM :



FOR SOME c IN (a, b) ,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

THERE IS A SIMILAR MEAN VALUE THEOREM (FOR INTEGRALS) :



FOR SOME x^* IN $[a, b]$,

$$\int_a^b f(x) dx = f(x^*) (b - a)$$

EXAMPLE : $f(x) = x^2$ IS CONTINUOUS ON $[1, 4]$

$$\int_1^4 x^2 dx = \frac{1}{3} x^3 \Big|_1^4 = \frac{1}{3} (4^3 - 1^3) = \frac{63}{3} = 21$$

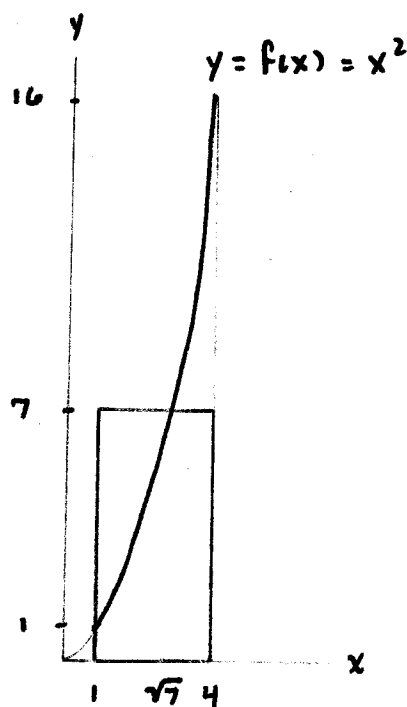
ACCORDING TO THE M.V.T. THERE SHOULD BE SOME x^* IN $[1, 4]$ AT WHICH

$$\int_1^4 x^2 dx = (x^*)^2 (4-1)$$

$$21 = 3(x^*)^2$$

$$7 = (x^*)^2$$

THE SOLUTION IN $[1, 4]$ IS $x^* = \sqrt{7} \approx 2.65$



NOW WE RETURN TO

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

ODDLY ENOUGH, FUNCTIONS OF THE TYPE $\int_a^x f(t) dt$ DO ARISE OFTEN IN APPLICATIONS, E.G., THE FRESNEL FUNCTION

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

IS IMPORTANT IN PHYSICS (OPTICS).

SINCE $\sin\left(\frac{\pi t^2}{2}\right)$ HAS NO ELEMENTARY ANTIDERIVATIVE, $S(x)$ REALLY CANNOT BE SIMPLIFIED ANY FURTHER.

HOWEVER, ITS DERIVATIVES ARE EASY:

$$\begin{aligned} S'(x) &= \frac{d}{dx} S(x) = \frac{d}{dx} \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \\ &= \sin\left(\frac{\pi x^2}{2}\right) \end{aligned}$$

$$S''(x) = \left(\sin\left(\frac{\pi x^2}{2}\right)\right)' = \pi x \left(\frac{\pi x^2}{2}\right)'$$

ETC.

HOWEVER, WHAT ABOUT THE DERIVATIVES OF

$$S(x^2) = \int_0^{x^2} \sin\left(\frac{\pi t^2}{2}\right) dt$$

OR

$$S(e^x) = \int_0^{e^x} \sin\left(\frac{\pi t^2}{2}\right) dt$$

ETC. ?

FOR THIS SORT OF THING WE NEED TO COMBINE OUR NEW DIFFERENTIATION FORMULA

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

WITH THE CHAIN RULE

$$\frac{d}{dx} F(u(x)) = F'(u(x))u'(x)$$

TO GET

$$\frac{d}{dx} \int_a^{u(x)} f(t) dt = f(u(x))u'(x)$$

EXAMPLES :

$$\begin{aligned} 1. \quad \frac{d}{dx} \int_3^{x^3} t^2 \sin t dt &= (x^3)^2 \sin(x^3) (x^3)' \\ &= x^4 \sin(x^3) (3x^2) = 3x^6 \sin(x^3) \end{aligned}$$

$$\begin{aligned} 2. \quad \text{FIND } y' \text{ IF } y &= \int_{e^x}^0 \sin^3 t dt \\ y' &= \frac{d}{dx} \int_{e^x}^0 \sin^3 t dt = - \frac{d}{dx} \int_0^{e^x} \sin^3 t dt = - \sin^3(e^x) (e^x)' \\ &= - e^x \sin^3(e^x) \end{aligned}$$

$$3. \quad \frac{d}{dx} \int_{\cos x}^{5x} \cos(t^2) dt = \frac{d}{dx} \left[\int_{\cos x}^0 \cos(t^2) dt + \int_0^{5x} \cos(t^2) dt \right] =$$

$$\frac{d}{dx} \left[- \int_0^{\cos x} \cos(t^2) dt + \int_0^{5x} \cos(t^2) dt \right] =$$

$$\begin{aligned} &- \cos(\cos^2 x) (-\sin x) + \cos((5x)^2) (5) = \\ &\cos(\cos^2 x) \sin x + 5 \cos(25x^2) \end{aligned}$$