

DIFFERENTIATION AND INTEGRATION OF SERIES

TWO THINGS WE KNOW :

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots$$

$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots$$

FOR ALL x .

NOW NOTICE THAT

$$(x)' = 1$$

$$\left(\frac{1}{3!} x^3\right)' = \frac{1}{3!} (3x^2) = \frac{1}{2!} x^2$$

$$\left(\frac{1}{5!} x^5\right)' = \frac{1}{5!} (5x^4) = \frac{1}{4!} x^4$$

$$\left(\frac{1}{7!} x^7\right)' = \frac{1}{7!} (7x^6) = \frac{1}{6!} x^6$$

⋮

SO

$$(x)' - \left(\frac{1}{3!} x^3\right)' + \left(\frac{1}{5!} x^5\right)' - \left(\frac{1}{7!} x^7\right)' + \dots = \cos x$$

$$= (\sin x)'$$

THE DERIVATIVE OF $\sin x$ CAN BE COMPUTED BY DIFFERENTIATING
ITS MACLAURIN SERIES TERM-BY-TERM JUST AS IF IT WERE
A POLYNOMIAL.

THEOREM : IF

$$f(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots \\ + c_k(x-x_0)^k + \dots$$

ON THE OPEN INTERVAL $(x_0 - R, x_0 + R)$, THEN $f(x)$ IS DIFFERENTIABLE ON THIS INTERVAL AND

$$f'(x) = (c_0)' + (c_1(x-x_0))' + (c_2(x-x_0)^2)' + \dots \\ + (c_k(x-x_0)^k)' + \dots \\ = c_1 + 2c_2(x-x_0) + \dots + k c_k (x-x_0)^{k-1} + \dots \\ = \sum_{k=1}^{\infty} k c_k (x-x_0)^{k-1}$$

HIGHER ORDER DERIVATIVES ARE COMPUTED IN THE SAME WAY.

EXAMPLE : $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ FOR ALL x

$$(e^x)' = \sum_{k=1}^{\infty} k \frac{1}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} \\ = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots \\ = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \\ = e^x$$

NOTE ON PREVIOUS EXAMPLE : DESPITE APPEARANCES THE TWO

SERIES $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ AND $\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}$ ARE EXACTLY

THE SAME, AS BECOMES APPARENT WHEN YOU START WRITING THE TERMS OUT. EACH CAN BE OBTAINED FROM THE OTHER BY RE-INDEXING, E.G.,

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} \quad ; \quad \text{LET } n = k-1. \text{ THEN}$$

$$k=1 \Rightarrow n=1-1=0$$

$$\text{SO } \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\text{WHICH IS THE SAME AS } \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

ANOTHER EXAMPLE :

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \quad \Rightarrow \quad f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$\text{LET } n = k-1. \text{ THEN } k = n+1$$

$$\text{AND } k=1 \Rightarrow n=1-1=0 \text{ SO}$$

$$f'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

(THIS WILL BE USEFUL WHEN WE RETURN TO DIFFERENTIAL EQUATIONS.)

POWER SERIES CAN ALSO BE INTEGRATED TERM-BY-TERM :

THEOREM : IF

$$f(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k$$

ON (x_0-R, x_0+R) , THEN

$$\int f(x) dx = \sum_{k=0}^{\infty} \frac{c_k}{k+1} (x-x_0)^{k+1} + C$$

ON (x_0-R, x_0+R) .

EXAMPLES :

$$\begin{aligned} 1. \quad \int \cos x \, dx &= \int \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \, dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)! (2k+1)} x^{2k+1} + C \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + C \\ &= \sin x + C \end{aligned}$$

2. WE OBTAIN A SERIES EXPANSION FOR $\arctan x$ ON $(-1, 1)$ BY INTEGRATING AN EXPANSION FOR $\frac{1}{1+x^2}$ AS FOLLOWS :

FOR $-1 < x < 1$,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad (\text{GEOMETRIC SERIES})$$

SO

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k \\ &= \sum_{k=0}^{\infty} (-1)^k x^k \end{aligned}$$

AND

$$\begin{aligned} \frac{1}{1+x^2} &= \sum_{k=0}^{\infty} (-1)^k (x^2)^k \\ &= \sum_{k=0}^{\infty} (-1)^k x^{2k} \end{aligned}$$

NOW INTEGRATE

$$\int \frac{1}{1+x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C$$

FOR SOME CHOICE OF C THIS GIVES THE ANTIDERIVATIVE $\text{ARCTAN } x$

FOR $\frac{1}{1+x^2}$. TO SOLVE

$$\text{ARCTAN } x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C$$

FOR C , PLUG IN $x=0$.

$$\text{ARCTAN } 0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot 0^{2k+1} + C$$

$$0 = 0 + C$$

SO

$$\text{ARCTAN } x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \quad \text{ON } (-1, 1)$$

NOTE : THIS SERIES ACTUALLY CONVERGES AT $x = \pm 1$ AS WELL (ALTERNATING SERIES TEST). IT'S NOT SO OBVIOUS (BUT IT'S TRUE) THAT THE SERIES ALSO CONVERGES TO $\text{ARCTAN } x$ FOR THESE TWO VALUES OF x , E.G.,

$$\text{ARCTAN}(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot 1^{2k+1}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

3. THE FUNCTION e^{-x^2} HAS NO ELEMENTARY ANTIDERIVATIVE SO WE HAVE NO MEANS OF CALCULATING $\int_0^1 e^{-x^2} dx$ EXACTLY. WE FIND A SERIES EXPANSION FOR THE VALUE AND APPROXIMATE IT BY A PARTIAL SUM AS FOLLOWS.

FOR ALL x ,

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

SO

$$\begin{aligned} e^{-x^2} &= \sum_{k=0}^{\infty} \frac{1}{k!} (-x^2)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} \end{aligned}$$

NOW INTEGRATE

$$\begin{aligned} \int e^{-x^2} dx &= \int \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} x^{2k+1} + C \end{aligned}$$

THUS,

$$\int_0^1 e^{-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}$$

$$= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \dots$$

SINCE THE SERIES IS ALTERNATING IT IS EASY TO APPROXIMATE THE ERROR MADE IN APPROXIMATING $\int_0^1 e^{-x^2} dx$ BY THE n^{TH} PARTIAL SUM

$$S_n = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \dots + \frac{(-1)^n}{(2n+1)n!}$$

$$\left| \int_0^1 e^{-x^2} dx - S_n \right| < \frac{1}{(2(n+1)+1)(n+1)!} = \frac{1}{(2n+3)(n+1)!}$$

FOR EXAMPLE, IF WE APPROXIMATE

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} \approx 0.743$$

THEN THE ERROR IS LESS THAN

$$\frac{1}{9 \cdot 4!} \approx 0.0046$$

THERE ARE A FEW MORE ALGEBRAIC TECHNIQUES FOR GETTING SERIES EXPANSIONS (OR AT LEAST PARTIAL SUMS OF THEM) QUICKLY. I'LL ILLUSTRATE WITH A FEW EXAMPLES.

EXAMPLES :

$$1. f(x) = x^2 \sin x : \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\begin{aligned} f(x) = x^2 \sin x &= x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+3} \end{aligned}$$

$$2. f(x) = \cos^2 x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x = \frac{1}{2} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2x)^{2k}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!} x^{2k}$$

$$= \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} x^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} x^{2k}$$

$$3. f(x) = e^{-x^2} \arctan x$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} \right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \right)$$

$$= \left(1 - x^2 + \frac{1}{2} x^4 - \dots \right) \left(x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \dots \right)$$

AT THIS POINT WE JUST START MULTIPLYING OUT, KEEPING TRACK OF WHICH PRODUCTS GIVE RISE TO WHICH POWERS OF x :

CONSTANT TERM : NONE
 x : $(1)(x) = x$
 x^2 : NONE
 x^3 : $(1)(-\frac{1}{3}x^3) + (-x^2)(x) = -\frac{4}{3}x^3$
 x^4 : NONE
 x^5 : $(1)(\frac{1}{5}x^5) + (-x^2)(-\frac{1}{3}x^3) + (\frac{1}{2}x^4)(x) =$
 $\frac{1}{5}x^5 + \frac{1}{3}x^5 + \frac{1}{2}x^5 = \frac{31}{30}x^5$
 \vdots

$$e^{-x^2} \text{ARCTAN } x = x - \frac{4}{3}x^3 + \frac{31}{30}x^5 + \dots$$

4. $f(x) = \tan x = \frac{\sin x}{\cos x}$

$$= \frac{\sum_{h=0}^{\infty} \frac{(-1)^h}{(2h+1)!} x^{2h+1}}{\sum_{h=0}^{\infty} \frac{(-1)^h}{(2h)!} x^{2h}} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots}$$

HERE WE JUST DO A LONG DIVISION (VERY LONG DIVISION) :

$$\begin{array}{r}
 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots \\
 \hline
 x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \\
 \hline
 x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots \\
 \hline
 x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \dots \\
 \hline
 \frac{1}{3}x^3 - \frac{1}{30}x^5 - \dots \\
 \hline
 \frac{1}{3}x^3 - \frac{1}{6}x^5 + \dots \\
 \hline
 \frac{2}{15}x^5 + \dots \\
 \hline
 \frac{2}{15}x^5 + \dots \\
 \hline
 \end{array}$$

ARITHMETIC: $-\frac{1}{6}x^3 + \frac{1}{2}x^3 = -\frac{1}{6}x^3 + \frac{3}{6}x^3 = \frac{2}{6}x^3 = \frac{1}{3}x^3$

$\frac{1}{120}x^5 - \frac{1}{24}x^5 = \frac{1}{120}x^5 - \frac{5}{120}x^5 = -\frac{4}{120}x^5 = -\frac{1}{30}x^5$

$-\frac{1}{30}x^5 + \frac{1}{6}x^5 = -\frac{1}{30}x^5 + \frac{5}{30}x^5 = \frac{4}{30}x^5 = \frac{2}{15}x^5$

so

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

NOTE: JUST TO SEE THE ADVANTAGES OF THE LONG DIVISION YOU SHOULD START COMPUTING DERIVATIVES OF $f(x) = \tan x$ AND TRY TO USE THE MACLAURIN FORMULAS $\frac{f^{(k)}(0)}{k!}$ TO GET THE COEFFICIENTS.