DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

RECALL:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$$

NEW STUFF : DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left[-\sin x \frac{1 - \cos h}{h} + \cos x \frac{\sin h}{h} \right]$$

$$= (-\sin x)(0) + (\cos x)(1)$$

$$\boxed{(\sin x)' = \cos x}$$

SIMILARLY, USING $\cos(x+h) = \cos x \cos h - \sin x \sin h$,

$$\boxed{(\cos x)' = -\sin x}$$

DO THIS ONE FOR YOURSELF !

NEXT,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2$$

$$= \sec^2 x$$

$$\boxed{(\tan x)' = \sec^2 x}$$

SIMILARLY, $\boxed{(\cot x)' = -\csc^2 x}$

DO THIS ONE FOR YOURSELF !

FINALLY,

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{(\cos x)(1)' - (1)(\cos x)'}{\cos^2 x}$$

$$= \frac{(\cos x)(0) - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\,\frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

$$\boxed{(\sec x)' = \sec x \tan x}$$

SIMILARLY, $\boxed{(\csc x)' = -\csc x \cot x}$

DO THIS ONE FOR YOURSELF !

EXAMPLES :

1. COMPUTE $\frac{dy}{dx}$ IF $y = x^3 \sin x$.

$$\frac{dy}{dx} = (x^3 \sin x)' = x^3 (\sin x)' + (\sin x)(x^3)'$$
$$= x^3 \cos x + (\sin x)(3x^2)$$
$$= x^3 \cos x + 3x^2 \sin x \quad = \quad x^2 (x \cos x + 3 \sin x)$$

2. COMPUTE $\frac{d^2 s}{dt^2}$ IF $s = \csc t$

$$\frac{ds}{dt} = (\csc t)' = -\csc t \cot t$$

$$\frac{d^2 s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = (-\csc t \cot t)' = (-\csc t)(\cot t)' + (\cot t)(-\csc t)'$$
$$= (-\csc t)(-\csc^2 t) + (\cot t)(\csc t \cot t)$$
$$= \csc^3 t + \csc t \cot^2 t \quad = \quad \csc t \,(\csc^2 t + \cot^2 t)$$

3. FIND THE EQUATION OF THE TANGENT LINE TO THE GRAPH OF $y = f(x) = \frac{\sin x}{x}$ AT $x = \pi$.

$$y - y_0 = m\,(x - x_0)$$

$x_0 = \pi$

$y_0 = f(\pi) = \frac{\sin \pi}{\pi} = 0$

$m = f'(\pi)$

COMPUTE $f'(x)$:

$$f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{x(\sin x)' - (\sin x)(x)'}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

THUS,

$$f'(\pi) = \frac{\pi \cos \pi - \sin \pi}{\pi^2} = \frac{\pi(-1) - 0}{\pi^2} = -\frac{1}{\pi} = m$$

SO

$$y - 0 = -\frac{1}{\pi}(x - \pi)$$

$$y = -\frac{1}{\pi}x + 1$$

4. FIND ALL VALUES OF x IN $[0, 2\pi]$ AT WHICH $f(x) = \sec x$ HAS A HORIZONTAL TANGENT LINE.

I.E., FIND ALL x IN $[0, 2\pi]$ AT WHICH

$$f'(x) = 0$$

$$\sec x \tan x = 0$$

$$\sec x = 0 \qquad \tan x = 0$$

$$\frac{1}{\cos x} = 0 \qquad \frac{\sin x}{\cos x} = 0$$

NO SOLUTIONS

$$\sin x = 0$$

$$x = 0, \pi, 2\pi$$

5. SHOW THAT $y = x\cos x$ IS A SOLUTION TO THE "DIFFERENTIAL EQUATION"

$$y'' + y = -2\sin x$$

$$y = x\cos x \implies y' = x(-\sin x) + (\cos x)(1)$$

$$= -x\sin x + \cos x$$

$$\implies y'' = -x(\cos x) + (\sin x)(-1) - \sin x$$

$$= -x\cos x - \sin x - \sin x$$

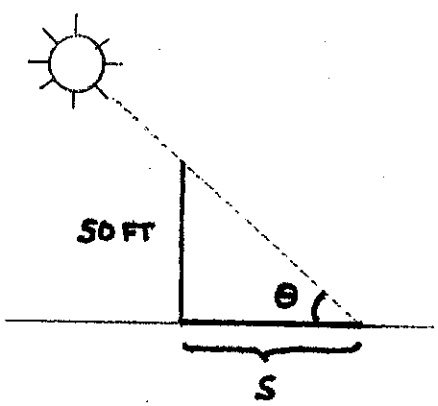
$$= -x\cos x - 2\sin x$$

SO

$$y'' + y = (-x\cos x - 2\sin x) + x\cos x$$

$$= -2\sin x$$

6. A 50 FT FLAGPOLE CASTS A SHADOW THAT CHANGES WITH THE ANGLE OF ELEVATION OF THE SUN. FIND THE RATE AT WHICH THE LENGTH S OF THE SHADOW IS CHANGING WITH THE ANGLE θ WHEN $\theta = 45°$.



WITH θ MEASURED IN RADIANS,

$$S = 50\cot\theta$$

$$\frac{ds}{d\theta} = -50\csc^2\theta$$

$$\left.\frac{ds}{d\theta}\right|_{\theta = \frac{\pi}{4}} = -50\csc^2\frac{\pi}{4}$$

$$= -50(2) = -100 \text{ FT/RAD}$$

IN FT/DEG THIS IS

$$\frac{-100 \text{ FT}}{\text{RAD}} \cdot \frac{\pi \text{ RAD}}{180 \text{ DEG}} = -\frac{5\pi}{9} \text{ FT/DEG} \approx -1.75 \text{ FT/DEG}$$