

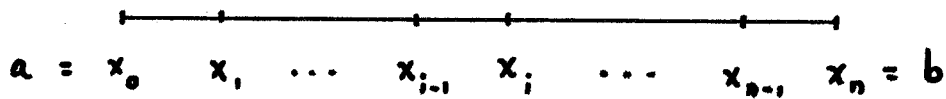
DEFINITE INTEGRALS

RECALL : RIEMANN SUM PROCEDURE

LET $f(x)$ BE CONTINUOUS ON $[a, b]$.

1. SUBDIVIDE $[a, b]$ INTO n SUBINTERVALS WITH ENDPONTS

$$a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_{n-1} < x_n = b$$



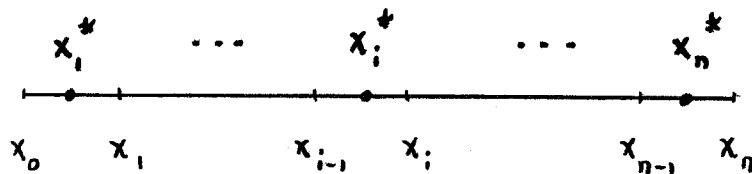
FOR EACH $i = 1, \dots, n$ LET

$$\Delta x_i = x_i - x_{i-1}$$

AND LET

$$\Delta x_{\max} = \text{LARGEST } \Delta x_i$$

2. INSIDE EACH $[x_{i-1}, x_i]$ SELECT A POINT x_i^*



EVALUATE

$$f(x_1^*), \dots, f(x_i^*), \dots, f(x_n^*)$$

AND COMPUTE

$$f(x_1^*) \Delta x_1, \dots, f(x_i^*) \Delta x_i, \dots, f(x_n^*) \Delta x_n$$

3. FORN THE RIEMANN SUM

$$f(x_1^*) \Delta x_1 + \dots + f(x_n^*) \Delta x_n = \sum_{i=1}^n f(x_i^*) \Delta x_i$$

4. REPEAT # 1-3 OVER AND OVER WITH SMALLER AND SMALLER Δx_{\max} AND TAKE THE LIMIT

$$\lim_{\Delta x_{\max} \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

IF $f(x) \geq 0$ ON $[a, b]$, THIS LIMIT CAN BE THOUGHT OF AS THE AREA UNDER THE GRAPH OF f AND ABOVE $[a, b]$.

OTHERWISE, IT CAN BE THOUGHT OF AS THE NET SIGNED AREA BETWEEN THE GRAPH OF f AND $[a, b]$.

HOWEVER, THIS LIMIT CAN ALSO BE THOUGHT OF IN MANY OTHER WAYS AS WELL.

E.G., WE WILL SEE LATER THAT IF THE INTERVAL $[a, b]$ IS ACTUALLY A METAL WIRE WHOSE

DENSITY AT ANY x IN $[a, b]$ IS $f(x)$, THEN

$\lim_{\Delta x_{\max} \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$ IS THE MASS OF THE WIRE.

THE POINT IS THAT THIS LIMIT, BECAUSE IT MEANS MANY DIFFERENT THINGS IN DIFFERENT CONTEXTS, DESERVES A NAME AND A SYMBOL.

$\lim_{\Delta x_{\max} \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$; , WHEN THE LIMIT EXISTS, IS CALLED

THE DEFINITE INTEGRAL OF $f(x)$ OVER $[a, b]$ AND IS DENOTED

$$\int_a^b f(x) dx$$

a IS CALLED THE LOWER, AND b THE UPPER LIMIT OF INTEGRATION.

BE CAREFUL NOT TO CONFUSE $\int_a^b f(x) dx$

AND $\int f(x) dx$. THEY ARE RELATED IN A

WAY WE WILL DESCRIBE NEXT TIME, BUT

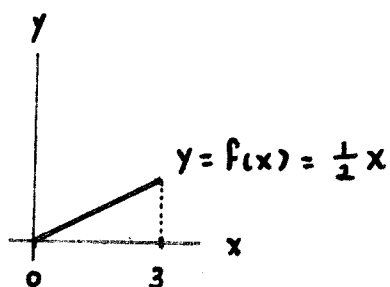
ARE ENTIRELY DIFFERENT TYPES OF THINGS.

THE FIRST IS A NUMBER. THE SECOND IS A

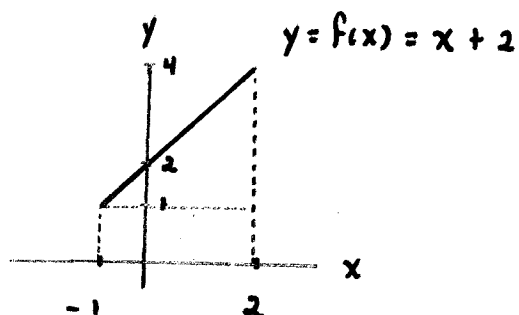
COLLECTION OF FUNCTIONS.

WE WILL NEED METHODS FOR EVALUATING THE NUMBERS $\int_a^b f(x) dx$
OTHER THAN COMPUTING THE LIMIT THAT DEFINES THEM.

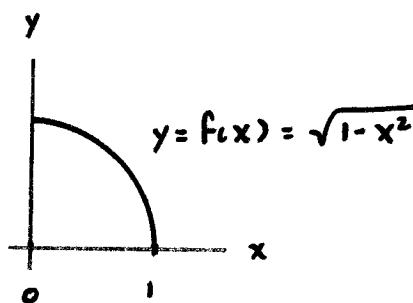
THESE METHODS GENERALLY INVOLVE ANTIDIFFERENTIATION, BUT SOME DEFINITE INTEGRALS CAN BE EVALUATED BY THINKING OF THEM AS AREAS, E.G.,



$$\int_0^3 \frac{1}{2}x dx = \frac{1}{2}(3)\left(\frac{1}{2} \cdot 3\right) = \frac{9}{4}$$



$$\int_{-1}^2 (x+2) dx = \frac{1}{2}(3)(3) + (3)(1) = \frac{9}{2} + 3 = \frac{15}{2}$$



$$\int_0^1 \sqrt{1-x^2} dx = \frac{1}{4}(\pi \cdot 1^2) = \frac{\pi}{4}$$

A FEW DEFINITIONS AND PROPERTIES :

DEFINITION 1 : $\int_a^a f(x) dx = 0$

DEFINITION 2 : $\int_b^a f(x) dx = - \int_a^b f(x) dx$

E.G., $\int_1^0 \sqrt{1-x^2} dx = - \frac{\pi}{4}$

PROPERTY 1 : $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

(LARGER SUMS ALSO)

PROPERTY 2 : $\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$

(LARGER DIFFERENCES ALSO)

PROPERTY 3 : FOR ANY CONSTANT c ,

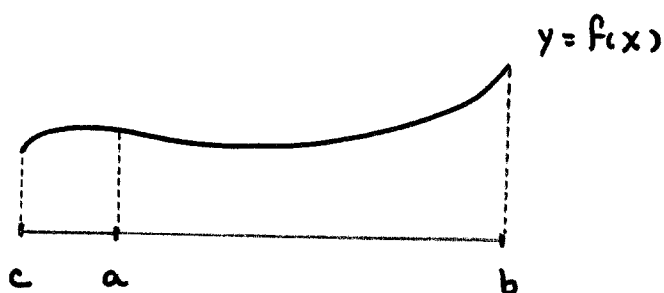
$$\int_a^b c f(x) dx = c \int_a^b f(x) dx$$

E.G., $\int_0^1 7\sqrt{1-x^2} dx = 7 \int_0^1 \sqrt{1-x^2} dx = 7\left(\frac{\pi}{4}\right) = \frac{7\pi}{4}$

PROPERTY 4 : $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

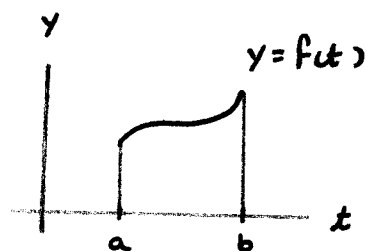
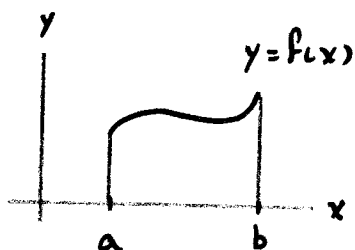
FOR ANY CHOICE OF c .

NOTE : PROPERTY 4 IS VALID EVEN IF c IS NOT IN $[a, b]$, E.G.,



$$\begin{aligned} \int_a^b &= \int_c^b - \int_c^a = \int_c^b + \int_a^c \\ &= \int_a^c + \int_c^b \end{aligned}$$

NOTICE THAT, FOR A DEFINITE INTEGRAL, THE NAME OF THE VARIABLE IS IRRELEVANT :



$$\int_a^b f(x) dx = \int_a^b f(t) dt$$

FOR THIS REASON THE VARIABLE IN A DEFINITE INTEGRAL IS OFTEN REFERRED TO AS A DUMMY VARIABLE.

OF COURSE, THIS IS NOT TRUE FOR AN INDEFINITE INTEGRAL, E.G.,

$$\int x^3 dx = \frac{1}{4} x^4 + C$$

BUT

$$\int t^3 dt = \frac{1}{4} t^4 + C$$

OR, EVEN BETTER YET,

$$\int 1 dx = x + C$$

BUT

$$\int 1 dt = t + C$$