CHAIN RULES

RECALL:

$$y = \sin(\tan t) \Rightarrow \frac{dy}{dt} = \cos(\tan t)\sec^2 t$$

$$y = f(g(t)) \Rightarrow \frac{dy}{dt} = f'(g(t))\, g'(t)$$

REPHRASED:

$$\begin{aligned} y &= f(x) \\ x &= g(t) \end{aligned} \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

($\frac{dy}{dx}$ IS EVALUATED AT $x = g(t)$)

FOR FUNCTIONS OF MORE THAN ONE VARIABLE THE PRIME (′) IS NEVER USED AND CHAIN RULES ALL LOOK LIKE

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

AND REMEMBERED BY "CANCELLING DIFFERENTIALS", E.G.,

VERSION #1:

$$\begin{aligned} z &= f(x,y) \\ x &= x(t) \\ y &= y(t) \end{aligned} \Rightarrow z = f(x(t), y(t)) \text{ AND}$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

($\frac{\partial z}{\partial x}$ AND $\frac{\partial z}{\partial y}$ ARE EVALUATED AT $x = x(t)$, $y = y(t)$)

<u>EXAMPLE</u> : $z = f(x,y) = x \arctan(xy)$

$x = x(t) = t^2$

$y = y(t) = te^t$

WE WILL COMPUTE $\frac{dz}{dt}$ FROM THE CHAIN RULE.

<u>NOTE</u> : THIS IS FOR PURPOSES OF ILLUSTRATING THE CHAIN RULE ONLY. THE "RIGHT" WAY TO FIND $\frac{dz}{dt}$ IN THIS CASE IS TO MAKE THE SUBSTITUTIONS $x = t^2$, $y = te^t$ INTO $z = f(x,y) = x \arctan(xy)$ AND COMPUTE THE DERIVATIVE DIRECTLY, I.E.,

$$z = f(t^2, te^t) = t^2 \arctan(t^3 e^t)$$

$$\frac{dz}{dt} = t^2\left[\frac{1}{1+(t^3e^t)^2}(t^3e^t)'\right] + \arctan(t^3e^t)(2t)$$

$$= \frac{t^2(t^3e^t + 3t^2e^t)}{1+t^4e^{2t}} + 2t\arctan(t^3e^t)$$

$$= \frac{t^4e^t(t+3)}{1+t^4e^{2t}} + 2t\arctan(t^3e^t)$$

THE "REAL" REASON FOR THE CHAIN RULE WILL EMERGE SHORTLY.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(x\arctan(xy))(2t) + \frac{\partial}{\partial y}(x\arctan(xy))(te^t + e^t)$$

$$= \left[x\frac{1}{1+(xy)^2}(y) + \arctan(xy)\right](2t) +$$

$$\left[x\frac{1}{1+(xy)^2}(x)\right](te^t + e^t)$$

$$= \left[\frac{xy}{1+x^2y^2} + \arctan(xy) \right](2t) + \left[\frac{x^2}{1+x^2y^2} \right](te^t + e^t)$$

$$= \left[\frac{t^3e^t}{1+t^6e^{2t}} + \arctan(t^3e^t) \right](2t) + \left[\frac{t^4}{1+t^6e^{2t}} \right](te^t + e^t)$$

$$= 2t\arctan(t^3e^t) + \frac{2t^4e^t}{1+t^6e^{2t}} + \frac{t^5e^t + t^4e^t}{1+t^3e^{2t}}$$

$$= 2t\arctan(t^3e^t) + \frac{t^5e^t + 3t^4e^t}{1+t^6e^{2t}}$$

$$= 2t\arctan(t^3e^t) + \frac{t^4e^t(t+3)}{1+t^6e^{2t}}$$

GEOMETRICAL INTERPRETATION :

$z = f(x,y)$ DESCRIBES A SURFACE IN SPACE ABOVE THE XY-PLANE

$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ DESCRIBES A CURVE IN THE XY-PLANE

$z = f(x(t), y(t))$ IS THE RESTRICTION OF THE FUNCTION TO THE CURVE

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ GIVES THE RATE OF CHANGE OF THE FUNCTION "ALONG THE CURVE" IN TERMS OF THE RATES OF CHANGE OF z WITH RESPECT TO x AND y AND THE VELOCITY OF THE CURVE.

VERSION #2 :

$$z = f(x,y), \quad x = x(s,t), \quad y = y(s,t) \quad \Rightarrow \quad z = f(x(s,t), y(s,t)) \text{ AND}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

NOTE : THE PATTERN IS THE SAME FOR EVERY VERSION OF THE CHAIN RULE : WRITE DOWN THE DERIVATIVES OF THE FUNCTION WITH RESPECT TO EACH OF ITS ORIGINAL VARIABLES, "CANCEL" THE DIFFERENTIALS YOU DON'T WANT, PUT IN THE ONES YOU DO WANT, THEN ADD, E.G.,

$$A = J(p, q, r, s)$$
$$p = p(\alpha, \beta)$$
$$q = q(\alpha, \beta)$$
$$r = r(\alpha, \beta)$$
$$s = s(\alpha, \beta)$$

$$\Rightarrow \quad \frac{\partial A}{\partial \beta} = \frac{\partial A}{\partial p}\frac{\partial p}{\partial \beta} + \frac{\partial A}{\partial q}\frac{\partial q}{\partial \beta} + \frac{\partial A}{\partial r}\frac{\partial r}{\partial \beta} + \frac{\partial A}{\partial s}\frac{\partial s}{\partial \beta}$$

EXAMPLES :

1. $w = f(s)$ AND $s = ax + y$ (a IS A NONZERO CONSTANT)

WE SHOW THAT

$$\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = 0$$

$$\frac{\partial w}{\partial x} = \frac{dw}{ds}\frac{\partial s}{\partial x} = \frac{dw}{ds}a$$

$$\frac{\partial w}{\partial y} = \frac{dw}{ds}\frac{\partial s}{\partial y} = \frac{dw}{ds}\cdot 1 = \frac{dw}{ds}$$

THUS,

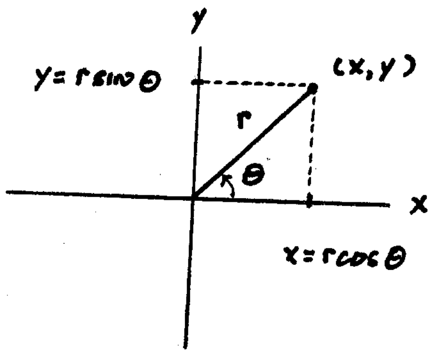
$$\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = \frac{dw}{ds}a - a\frac{dw}{ds} = 0$$

2. $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

WE SHOW THAT

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

NOTE : r AND θ ARE "POLAR COORDINATES" IN THE XY-PLANE



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}r^2\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + 0 + \left(\frac{\partial z}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

AS PROMISED.

* 3. $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

WE SHOW THAT

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

(CALLED THE <u>LAPLACIAN</u> OF $z = f(x,y)$)

FROM THE PREVIOUS EXAMPLE,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$= z_x \cos\theta + z_y \sin\theta$$

(THE SUBSCRIPTS WILL BE MORE CONVENIENT FOR THE REST OF THE CALCULATION)

NOW WE COMPUTE

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}(z_x \cos\theta + z_y \sin\theta)$$

$$= \frac{\partial}{\partial r}(z_x \cos\theta) + \frac{\partial}{\partial r}(z_y \sin\theta)$$

$$= \cos\theta \frac{\partial z_x}{\partial r} + \sin\theta \frac{\partial z_y}{\partial r}$$

$$= \cos\theta \left[\frac{\partial z_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z_x}{\partial y}\frac{\partial y}{\partial r} \right] + \sin\theta \left[\frac{\partial z_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z_y}{\partial y}\frac{\partial y}{\partial r} \right]$$

$$= \cos\theta \left[z_{xx}\cos\theta + z_{xy}\sin\theta \right] + \sin\theta \left[z_{yx}\cos\theta + z_{yy}\sin\theta \right]$$

$$= z_{xx}\cos^2\theta + z_{xy}\cos\theta\sin\theta + z_{yx}\cos\theta\sin\theta + z_{yy}\sin^2\theta$$

$$= z_{xx}\cos^2\theta + 2 z_{xy}\cos\theta\sin\theta + z_{yy}\sin^2\theta$$

$$\frac{\partial^2 z}{\partial r^2} = \cos^2\theta \frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta \frac{\partial^2 z}{\partial y \partial x} + \sin^2\theta \frac{\partial^2 z}{\partial y^2}$$

EXERCISE: SHOW THAT

$$\frac{\partial^2 z}{\partial \theta^2} = -r\cos\theta \frac{\partial z}{\partial x} - r\sin\theta \frac{\partial z}{\partial y} - 2r^2\cos\theta\sin\theta \frac{\partial^2 z}{\partial y \partial x} + r^2\sin^2\theta \frac{\partial^2 z}{\partial x^2} + r^2\cos^2\theta \frac{\partial^2 z}{\partial y^2}$$

AND THEN FINISH THE EXAMPLE.