THE CHAIN RULE :

MOTIVATION :

1. $y = (x^2+1)^3$

$= x^6 + 3x^4 + 3x^2 + 1$

$y' = 6x^5 + 12x^3 + 6x$

$= 6x(x^4 + 2x^2 + 1)$

$= 6x(x^2+1)^2$

$= 3(x^2+1)^2(2x)$

$= 3(x^2+1)^2(x^2+1)'$

$\vdots$

$$\boxed{\begin{aligned} y &= (u(x))^n \\ y' &= n(u(x))^{n-1}u'(x) \end{aligned}}$$

2. $y = \sin 2x$

$= 2\sin x \cos x$

$y' = 2(-\sin^2 x + \cos^2 x)$

$= 2(\cos^2 x - \sin^2 x)$

$= 2\cos 2x$

$= \cos 2x \ (2x)'$

$\vdots$

$$\boxed{\begin{aligned} y &= \sin u(x) \\ y' &= \cos u(x)\, u'(x) \end{aligned}}$$

BOTH ARE EXAMPLES OF THE "CHAIN RULE"

MORE EXAMPLES :

1. $y = \cos u(x) \quad \Rightarrow \quad y' = -\sin u(x)\, u'(x)$
2. $y = \tan u(x) \quad \Rightarrow \quad y' = \sec^2 u(x)\, u'(x)$
3. $y = \cot u(x) \quad \Rightarrow \quad y' = -\csc^2 u(x)\, u'(x)$
4. $y = \sec u(x) \quad \Rightarrow \quad y' = \sec u(x) \tan u(x)\, u'(x)$
5. $y = \csc u(x) \quad \Rightarrow \quad y' = -\csc u(x) \cot u(x)\, u'(x)$

THE GENERAL CHAIN RULE :

$$y = f(u(x)) \Rightarrow y' = f'(u(x))u'(x)$$

WE'LL LOOK AT SOME EXAMPLES FIRST AND THEN I'LL TRY TO GIVE YOU A REASON TO BELIEVE THAT THIS IS TRUE.

1. $y = 4\cos(x^3)$

$$y' = 4(\cos(x^3))' = 4(-\sin(x^3)(x^3)') = 4(-\sin(x^3)(3x^2))$$
$$= -12x^2\sin(x^3)$$

2. $y = (x^2 - x + 1)^{23}$

$$y' = 23(x^2-x+1)^{22}(x^2-x+1)' = 23(x^2-x+1)^{22}(2x-1)$$

3. $s = \sqrt{1+\tan t} = (1+\tan t)^{\frac{1}{2}}$

$$\frac{ds}{dt} = \frac{1}{2}(1+\tan t)^{-\frac{1}{2}}(1+\tan t)' = \frac{1}{2}(1+\tan t)^{-\frac{1}{2}}\sec^2 t$$
$$= \frac{\sec^2 t}{2\sqrt{1+\tan t}}$$

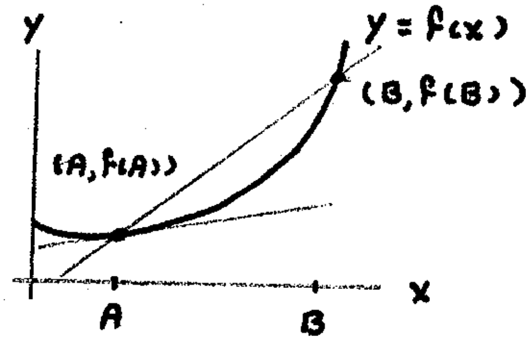
4. $R = \sec(\tan s)$

$$\frac{dR}{ds} = \sec(\tan s)\tan(\tan s)(\tan s)'$$
$$= \sec(\tan s)\tan(\tan s)\sec^2 s$$

WE'LL DO SOME MORE COMPLICATED EXAMPLES SHORTLY, BUT FIRST WE SHOULD TRY TO UNDERSTAND WHY IT'S TRUE.

A CAREFUL PROOF IS SURPRIZINGLY TRICKY SO I'LL JUST TRY TO GIVE YOU A REASON TO BELIEVE.

<u>NOTE</u> :



$$f'(A) = \lim_{B \to A} \frac{f(B)-f(A)}{B-A}$$

NOW, ASSUMING f AND u ARE DIFFERENTIABLE,

$$\begin{aligned}(f(u(x)))' &= \lim_{h\to 0} \frac{f(u(x+h)) - f(u(x))}{h} \\ &= \lim_{h\to 0} \frac{f(u(x+h)) - f(u(x))}{u(x+h) - u(x)} \; \frac{u(x+h)-u(x)}{h} \\ &= \lim_{h\to 0} \frac{f(\overset{B}{u(x+h)}) - f(\overset{A}{u(x)})}{\underset{B}{u(x+h)} - \underset{A}{u(x)}} \; \frac{u(x+h)-u(x)}{h} \\ &= f'(u(x))\, u'(x)\end{aligned}$$

BECAUSE u DIFFERENTIABLE $\Rightarrow$ u CONTINUOUS $\Rightarrow$ $u(x+h) \to u(x)$ AS $h \to 0$. (CAN YOU SEE WHERE THIS "PROOF" MIGHT RUN INTO TROUBLE?)

SOME MORE EXAMPLES :

1. $f(x) = \left(\frac{1+x^2}{1-x^2} \right)^{17}$

$$f'(x) = 17 \left(\frac{1+x^2}{1-x^2} \right)^{16} \left(\frac{1+x^2}{1-x^2} \right)'$$

$$= 17 \left(\frac{1+x^2}{1-x^2} \right)^{16} \left(\frac{(1-x^2)(2x) - (1+x^2)(-2x)}{(1-x^2)^2} \right)$$

$$= 17 \frac{(1+x^2)^{16}}{(1-x^2)^{16}} \left(\frac{4x}{(1-x^2)^2} \right) = \frac{68x(1+x^2)^{16}}{(1-x^2)^{18}}$$

2. FIND $\frac{d^2y}{dx^2}$ IF $y = x \tan\left(\frac{1}{x}\right)$

$$\frac{dy}{dx} = \left(x \tan\left(\tfrac{1}{x}\right)\right)' = x\left(\tan\left(\tfrac{1}{x}\right)\right)' + \tan\left(\tfrac{1}{x}\right)(x)'$$

$$= x \sec^2\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x}\right)' + \tan\left(\tfrac{1}{x}\right)$$

$$= x \sec^2\left(\tfrac{1}{x}\right)\left(-\tfrac{1}{x^2}\right) + \tan\left(\tfrac{1}{x}\right)$$

$$= -\left(\tfrac{1}{x}\right)\sec^2\left(\tfrac{1}{x}\right) + \tan\left(\tfrac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = -\left[\tfrac{1}{x}\left(\sec^2\left(\tfrac{1}{x}\right)\right)' + \sec^2\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x}\right)' \right] + \sec^2\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x}\right)'$$

THESE TWO CANCEL

$$= -\tfrac{1}{x}\left(2\sec\left(\tfrac{1}{x}\right)\left(\sec\left(\tfrac{1}{x}\right)\right)'\right)$$

$$= -\tfrac{2}{x}\sec\left(\tfrac{1}{x}\right)\sec\left(\tfrac{1}{x}\right)\tan\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x}\right)'$$

$$= -\tfrac{2}{x}\sec^2\left(\tfrac{1}{x}\right)\tan\left(\tfrac{1}{x}\right)\left(-\tfrac{1}{x^2}\right)$$

$$= \frac{2\sec^2\left(\tfrac{1}{x}\right)\tan\left(\tfrac{1}{x}\right)}{x^3}$$

3. FIND THE EQUATION OF THE TANGENT LINE TO THE GRAPH OF $y = f(x) = 3\cot^3 x$ AT $x = \frac{\pi}{4}$

$$y - y_0 = m\,(x - x_0)$$

$$\frac{\pi}{4}$$

$$f(\tfrac{\pi}{4}) = 3\cot^3 \tfrac{\pi}{4} = 3$$

$$f'(\tfrac{\pi}{4}) \;:\; f'(x) = 3\,(3\cot^2 x\,(\cot x)')$$

$$= 9\cot^2 x\,(-\csc^2 x)$$

$$= -9\cot^2 x \csc^2 x$$

$$f'(\tfrac{\pi}{4}) = -9\,(1)^2(\sqrt{2})^2 = -18$$

$$y - 3 = -18\,(x - \tfrac{\pi}{4})$$

$$y = -18x + \frac{9\pi}{2} + 3$$

CHAIN RULE REPHRASED :

$$y = f(u) \text{ AND } u = u(x)$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}$$

" LEIBNITZ NOTATION "