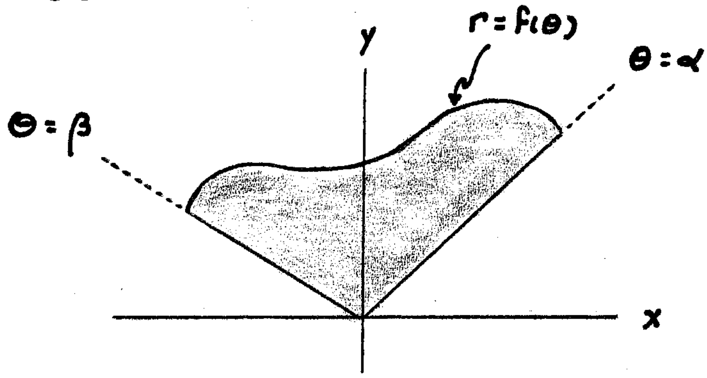
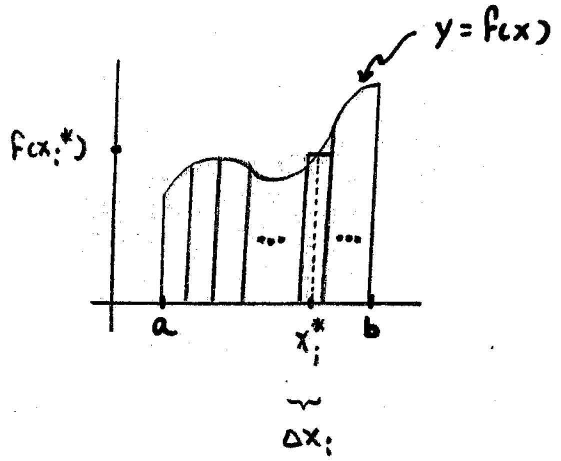
AREAS IN POLAR COORDINATES

CONSIDER A REGION IN THE PLANE OF THE FOLLOWING TYPE.



WE WANT TO COMPUTE ITS AREA.

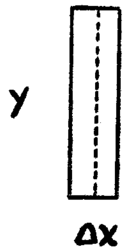
TO DO THIS WE NEED TO RECALL HOW WE ARRIVED AT OUR EARLIER FORMULAS FOR AREAS ("LIMITS OF RIEMANN SUMS")



$$\text{AREA} = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

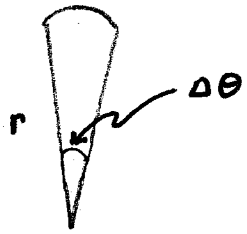
$$= \int_a^b f(x)\,dx$$

WE'LL DO THE SAME THING FOR THE POLAR REGION ABOVE, REPLACING RECTANGLES



AREA $= y\,\Delta x$

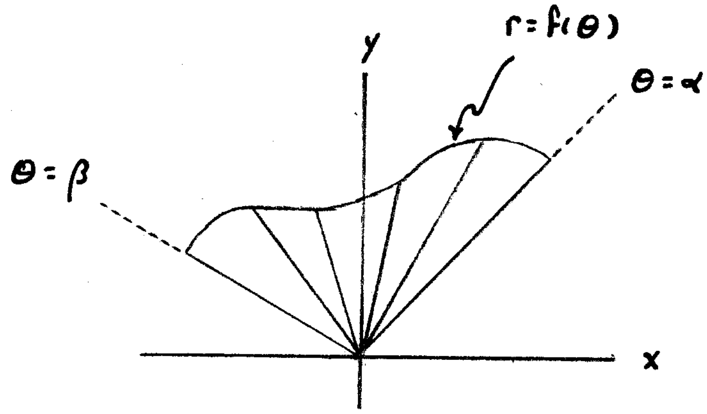
WITH CIRCULAR WEDGES



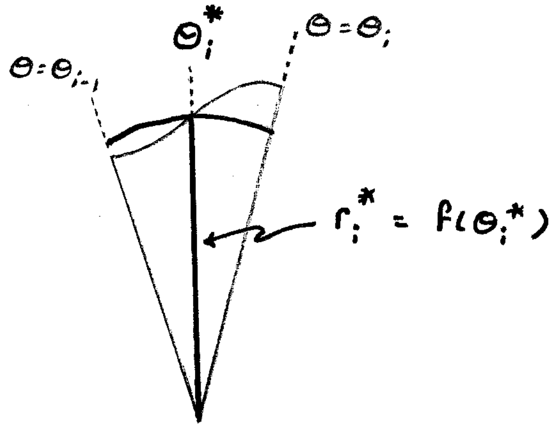
AREA $= \frac{1}{2} r^2 \Delta\theta$

THIS IS A FORMULA FROM TRIGONOMETRY. IF YOU DON'T REMEMBER IT, CHECK IT OUT WHEN $\Delta\theta = 2\pi$ AND THEN THINK ABOUT THE AREA YOU WOULD GET IF YOU USED ONLY $\Delta\theta$ OF A COMPLETE REVOLUTION.

NOW SUBDIVIDE THE POLAR REGION INTO SEGMENTS :



APPROXIMATE EACH SEGMENT BY A CIRCULAR WEDGE :



$$\text{AREA OF THE SEGMENT} \approx \text{AREA OF THE WEDGE}$$
$$\approx \frac{1}{2}(r_i^*)^2(\theta_i - \theta_{i-1})$$
$$\approx \frac{1}{2}(f(\theta_i^*))^2\,\Delta\theta_i$$

$$\text{AREA OF POLAR REGION} \approx \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2\,\Delta\theta_i$$

APPROXIMATIONS IMPROVE AS THE $\Delta\theta_i \to 0$ SO

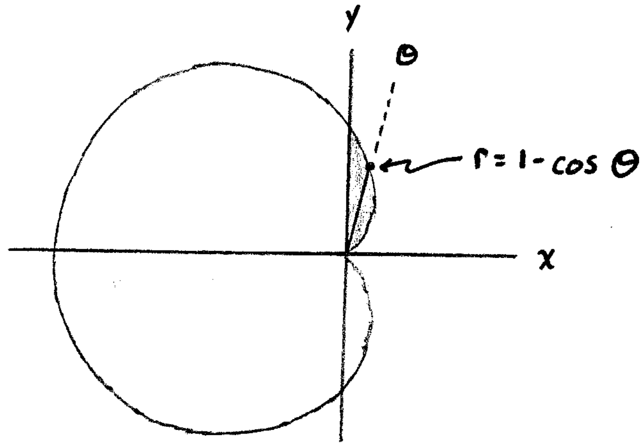
$$A = \lim_{\max \Delta\theta_i \to 0} \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i^*))^2\,\Delta\theta_i$$
$$= \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2\,d\theta$$

EXAMPLES :

1. AREA OF REGION IN THE FIRST QUADRANT INSIDE THE CARDIOID

$$r = 1 - \cos\theta$$

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{2}$	1
π	2



$0 \le \theta \le \frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}(f(\theta))^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta)\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta \Big|_0^{\frac{\pi}{2}} - 2\sin\theta \Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\sin 2\theta \Big|_0^{\frac{\pi}{2}} \right]$$

$$= \frac{1}{2}\left[\frac{3\pi}{4} - 2(1-0) + \frac{1}{4}(0-0) \right]$$

$$= \frac{3\pi}{8} - 1$$

<u>NOTE</u> : THE AREA OF THE ENTIRE CARDIOID CAN BE COMPUTED EITHER

AS

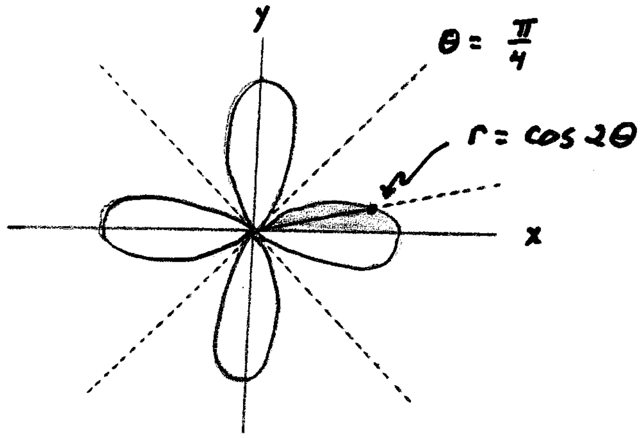
$$\int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2 d\theta$$

OR

$$2\int_0^{\pi} \frac{1}{2}(1-\cos\theta)^2 d\theta$$

USING SYMMETRY CAN OFTEN SIMPLIFY THE CALCULATIONS.

2. FIND THE AREA OF THE REGION ENCLOSED BY $r = \cos 2\theta$.

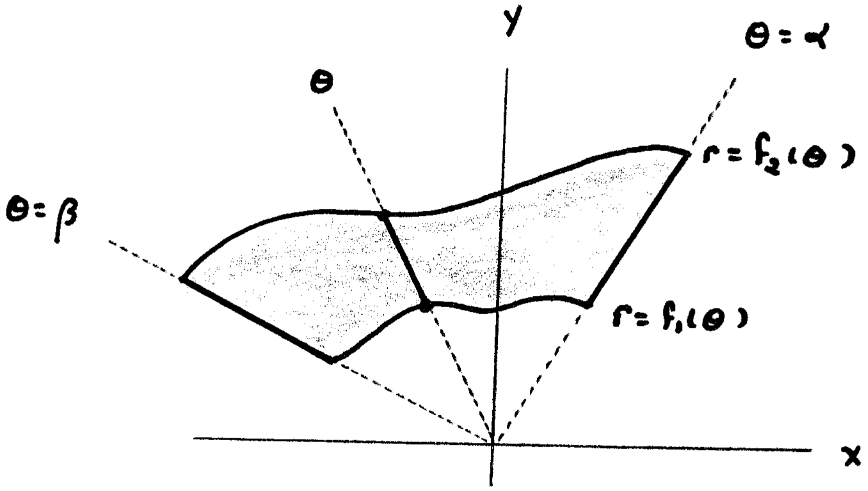


$$A = 8 \int_0^{\frac{\pi}{4}} \tfrac{1}{2} (\cos 2\theta)^2 d\theta = 4 \int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} (1 + \cos 4\theta) d\theta = 2\theta \Big|_0^{\frac{\pi}{4}} + \tfrac{1}{2} \sin 4\theta \Big|_0^{\frac{\pi}{4}}$$

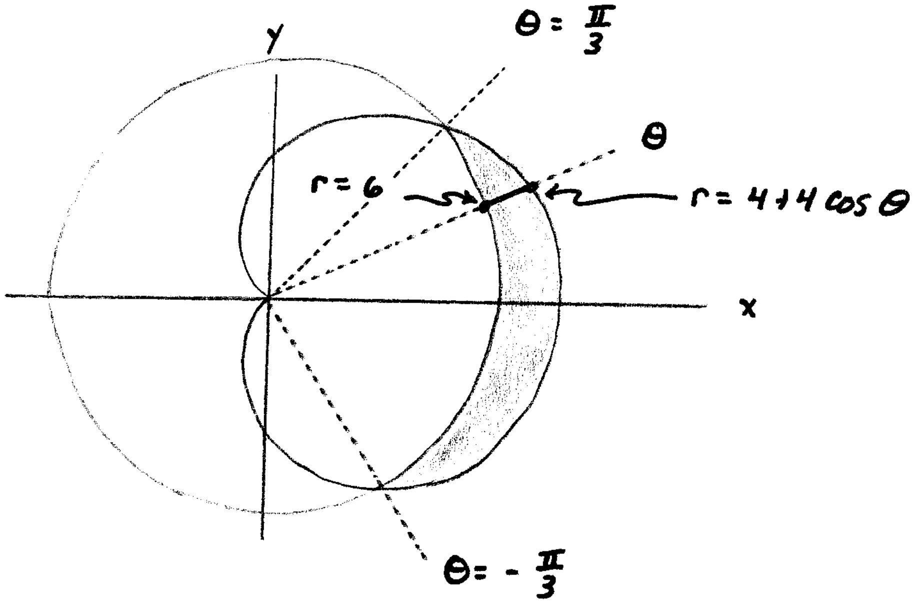
$$= \frac{\pi}{2}$$

A MORE GENERAL AREA FORMULA :



$$A = \int_\alpha^\beta \tfrac{1}{2} \left((f_2(\theta))^2 - (f_1(\theta))^2 \right) d\theta$$

<u>EXAMPLE</u> : FIND THE AREA OF THE REGION INSIDE THE CARDIOD $r = 4 + 4\cos\theta$ AND OUTSIDE THE CIRCLE $r = 6$.



<u>INTERSECTIONS</u> :

$$4 + 4\cos\theta = 6$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left((4+4\cos\theta)^2 - 6^2\right)d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (16 + 32\cos\theta + 16\cos^2\theta - 36)\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (16\cos\theta + 4 + 4\cos 2\theta - 10)\,d\theta$$

$$= 16\sin\theta\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} + 2\sin 2\theta\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} - 6\theta\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = 18\sqrt{3} - 4\pi$$