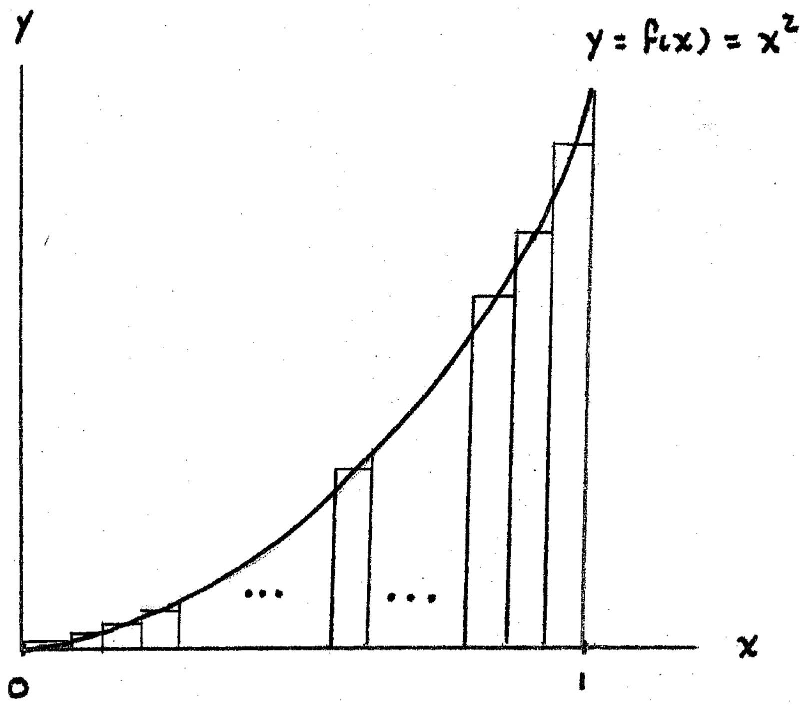
# AREAS AND RIEMANN SUMS

## n RECTANGLES




BASES : $[0, \frac{1}{n}]$ , $[\frac{1}{n}, \frac{2}{n}]$ , ... , $[\frac{n-1}{n}, 1]$

MIDPOINTS : $\frac{1}{2n}$ , $\frac{3}{2n}$ , ... , $\frac{2n-1}{2n}$

$$A \approx f\left(\frac{1}{2n}\right)\cdot\frac{1}{n} + f\left(\frac{3}{2n}\right)\cdot\frac{1}{n} + \cdots + f\left(\frac{2n-1}{2n}\right)\cdot\frac{1}{n} =$$

$$\left(\frac{1}{2n}\right)^2\cdot\frac{1}{n} + \left(\frac{3}{2n}\right)^2\cdot\frac{1}{n} + \cdots + \left(\frac{2n-1}{2n}\right)^2\cdot\frac{1}{n} =$$

$$\frac{1^2}{4n^2}\cdot\frac{1}{n} + \frac{3^2}{4n^2}\cdot\frac{1}{n} + \cdots + \frac{(2n-1)^2}{4n^2}\cdot\frac{1}{n} =$$

$$(1^2 + 3^2 + \cdots + (2n-1)^2)\cdot\frac{1}{4n^3}$$

E.G., IF $n = 5$,

$$A \approx (1^2+3^2+5^2+7^2+9^2)\cdot\frac{1}{4\cdot 5^3} = \frac{165}{500} = 0.3300$$

FACT: $1^2+3^2+\cdots+(2n-1)^2 = \frac{4n^3-n}{3}$

E.G., $n = 3$ : $1^2+3^2+5^2 = 1+9+25 = 35$

$$\frac{4\cdot 3^3-3}{3} = \frac{108-3}{3} = \frac{105}{3} = 35$$

( THIS IS PROVED BY "MATHEMATICAL INDUCTION" )

THUS,

$$A \approx (1^2+3^2+\cdots+(2n-1)^2)\frac{1}{4n^3} = \frac{4n^3-n}{3}\,\frac{1}{4n^3}$$

$$= \frac{4n^3-n}{12n^3}$$

AND THE APPROXIMATION IMPROVES AS $n \to \infty$, I.E.,

$$A = \lim_{n\to\infty}\frac{4n^3-n}{12n^3} = \lim_{n\to\infty}\frac{4-\frac{1}{n^2}}{12}$$

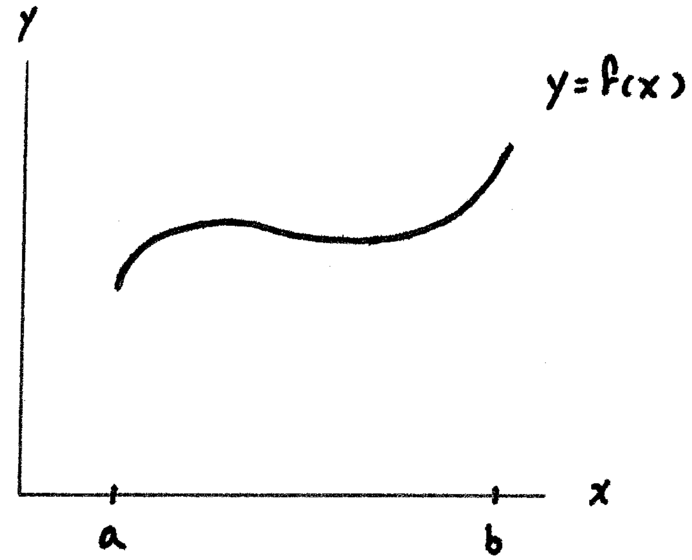
$$= \frac{4-0}{12}$$

$$= \frac{1}{3}$$

CONCLUSION: THE AREA UNDER THE GRAPH OF $f(x) = x^2$ FROM $x = 0$ TO $x = 1$ IS PRECISELY $\frac{1}{3}$.

THE GENERAL CONSTRUCTION:

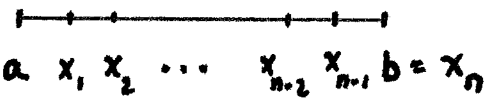
LET $f(x)$ BE CONTINUOUS AND NON-NEGATIVE ( $f(x) \geq 0$ ) ON $[a, b]$.




TO COMPUTE THE AREA UNDER THE GRAPH OF $f(x)$ AND ABOVE THE INTERVAL $[a, b]$ WE PROCEED AS FOLLOWS:

1. SUBDIVIDE THE INTEVAL $[a, b]$ INTO $n$ SUBINTERVALS WITH ENDPOINTS

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = b$$




FOR EACH $i = 1, 2, \ldots, n-1, n$, LET

$$\Delta x_i = x_i - x_{i-1} = \text{LENGTH OF } [x_{i-1}, x_i]$$

**NOTE :** IF ALL OF THE SUBINTERVALS HAVE THE SAME LENGTH WE DENOTE ALL OF THE $\Delta X_i$ SIMPLY $\Delta X$. OTHERWISE THE LARGEST OF THE $\Delta X_i$ WILL BE DENOTED $\Delta X_{max}$.

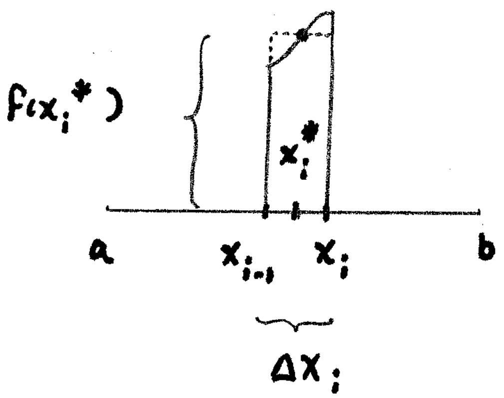
2. INSIDE EACH $[x_{i-1}, x_i]$ SELECT A POINT $x_i^*$


EVALUATE

$$f(x_1^*),\ f(x_2^*),\ \dots,\ f(x_{n-1}^*),\ f(x_n^*)$$

AND COMPUTE

$$f(x_1^*)\Delta x_1\ ,\ f(x_2^*)\Delta x_2\ ,\dots,\ f(x_{n-1}^*)\Delta x_{n-1}\ ,\ f(x_n^*)\Delta x_n$$



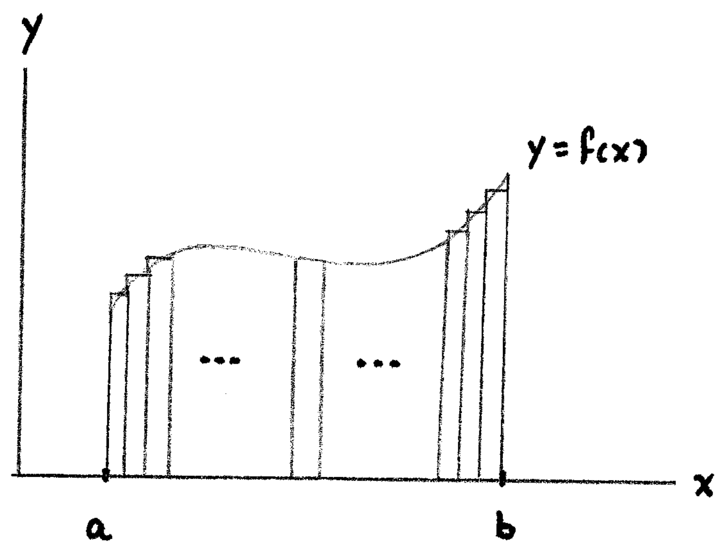

3. FORM THE <u>RIEMANN</u> <u>SUM</u> APPROXIMATION

$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_{n-1}^*)\Delta x_{n-1} + f(x_n^*)\Delta x_n$$
$$= \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

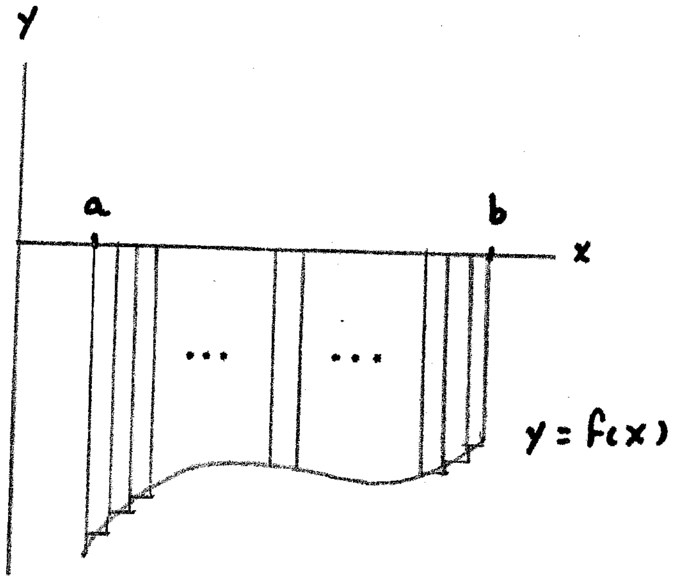
( <u>SIGMA</u> <u>NOTATION</u> : THE SUM OF ALL THE $f(x_i^*)\Delta x_i$ FOR $i$ TAKING VALUES FROM 1 TO $n$ )

4. REPEAT STEPS #1-3 OVER AND OVER WITH FINER AND FINER SUBDIVISIONS OF $[a, b]$ (I.E., SMALLER AND SMALLER $\Delta x_{max}$ ) AND TAKE THE LIMIT

$$\lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$



NOTICE THAT IF $f(x) \leq 0$ (RATHER THAN $f(x) \geq 0$) ON $[a,b]$, THEN THE RESULT OF THIS PROCEDURE WILL BE <u>MINUS</u> THE AREA BETWEEN THE GRAPH OF $f(x)$ AND $[a,b]$.




IF $f(x)$ TAKES BOTH POSITIVE AND NEGATIVE VALUES ON $[a,b]$, THEN THE PROCEDURE YIELDS THE <u>NET SIGNED AREA</u> BETWEEN THE GRAPH OF $f(x)$ AND THE INTERVAL $[a,b]$.