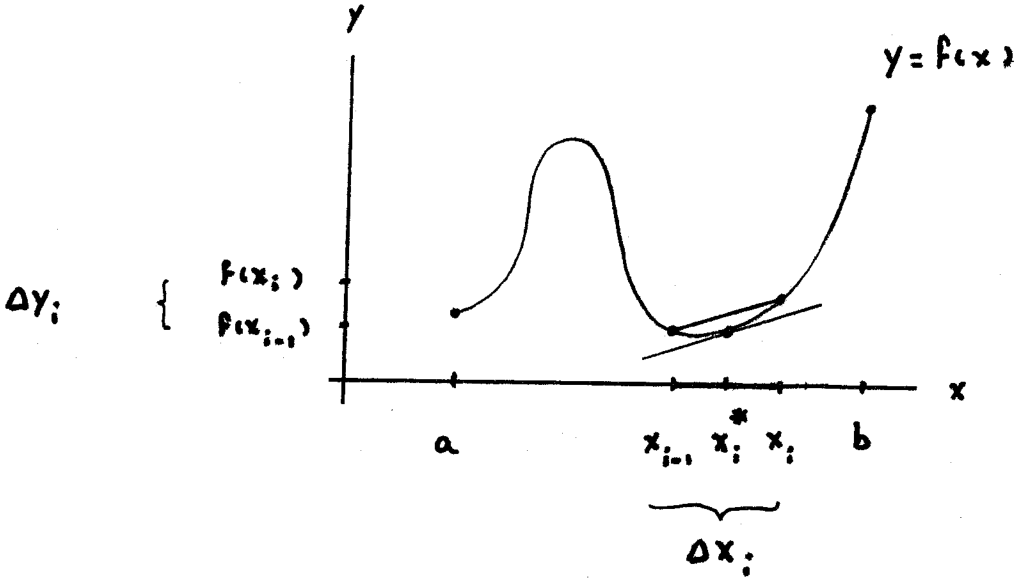
ARC LENGTH

LENGTHS OF CURVES IN THE PLANE



A SHORT ENOUGH SEGMENT IS NEARLY STRAIGHT SO ITS LENGTH CAN BE APPROXIMATED BY THE PYTHAGOREAN THEOREM.

$$\text{LENGTH OF GRAPH ABOVE } [x_{i-1}, x_i] \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

WHICH WE REWRITE AS FOLLOWS :

$$\begin{aligned}
\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} &= \sqrt{(\Delta x_i)^2 \left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right)} \\
&= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i \\
&= \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\ \Delta x_i \\
&= \sqrt{1 + (f'(x_i^*))^2}\ \Delta x_i
\end{aligned}$$

WHERE x_i^* IS THE POINT GUARANTEED BY THE MEAN VALUE THEOREM AT WHICH THE SLOPE OF THE TANGENT LINE $(f'(x_i^*))$ IS THE SAME AS THE SLOPE OF THE SECANT LINE JOINING THE ENDPOINTS OF THE GRAPH ABOVE $[x_{i-1}, x_i]$ $\left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)$.

THUS,

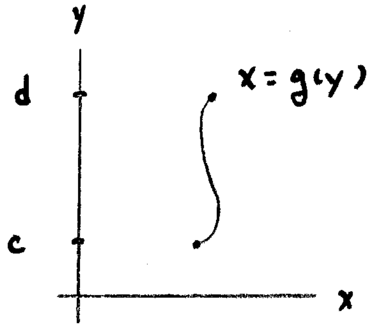
$$\text{LENGTH OF GRAPH} \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\, \Delta x_i$$

SO

$$\text{LENGTH OF GRAPH} = \lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\, \Delta x_i$$

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

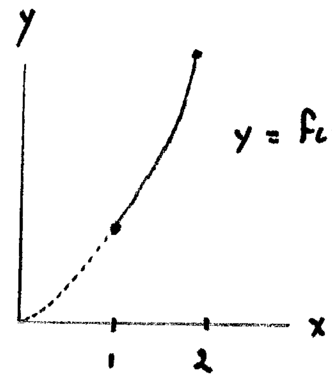
SIMILARLY FOR GRAPHS OF FUNCTIONS OF y :



$$\text{LENGTH} = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

EXAMPLES :

1. COMPUTE THE LENGTH OF THE CURVE $y = x^{\frac{3}{2}}$ OVER THE INTERVAL $[1, 2]$.



$$\text{LENGTH} = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \frac{3}{2} x^{\frac{1}{2}}$$

$$(f'(x))^2 = \frac{9}{4} x$$

$$= \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{4}{9} \int_{\frac{13}{4}}^{\frac{22}{4}} u^{\frac{1}{2}} du = \frac{8}{27} u^{\frac{3}{2}} \Big|_{\frac{13}{4}}^{\frac{22}{4}}$$

$$u = 1 + \frac{9}{4} x$$

$$du = \frac{9}{4} dx$$

$$x = 1 \Rightarrow u = \frac{13}{4}$$

$$x = 2 \Rightarrow u = \frac{22}{4}$$

$$= \frac{8}{27}\left[\left(\frac{22}{4}\right)^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}} \right]$$

$$= \frac{8}{27}\left(\frac{22\sqrt{22} - 13\sqrt{13}}{8} \right)$$

$$= \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$

2. COMPUTE THE LENGTH OF $x = g(y) = \frac{1}{8} y^4 + \frac{1}{4} y^{-2}$ OVER THE INTERVAL $1 \le y \le 4$.

$$\text{LENGTH} = \int_1^4 \sqrt{1 + (g'(y))^2}\, dy$$

FIRST WE'LL COMPUTE

$$g'(y) = \left(\tfrac{1}{8}y^4 + \tfrac{1}{4}y^{-2}\right)' = \tfrac{1}{2}y^3 - \tfrac{1}{2}y^{-3}$$

$$(g'(y))^2 = \left(\tfrac{1}{2}y^3 - \tfrac{1}{2}y^{-3}\right)^2 = \tfrac{1}{4}y^6 - 2\left(\tfrac{1}{2}y^3\right)\left(\tfrac{1}{2}y^{-3}\right) + \tfrac{1}{4}y^{-6}$$

$$= \tfrac{1}{4}y^6 - \tfrac{1}{2} + \tfrac{1}{4}y^{-6}$$

$$1 + (g'(y))^2 = \tfrac{1}{4}y^6 - \tfrac{1}{2} + \tfrac{1}{4}y^{-6} + 1$$

$$= \tfrac{1}{4}y^6 + \tfrac{1}{2} + \tfrac{1}{4}y^{-6}$$

$$= \left(\tfrac{1}{2}y^3 + \tfrac{1}{2}y^{-3}\right)^2$$

$$\sqrt{1 + (g'(y))^2} = \tfrac{1}{2}y^3 + \tfrac{1}{2}y^{-3}$$

$$\text{LENGTH} = \int_1^4 \tfrac{1}{2}y^3 + \tfrac{1}{2}y^{-3}\,dy$$

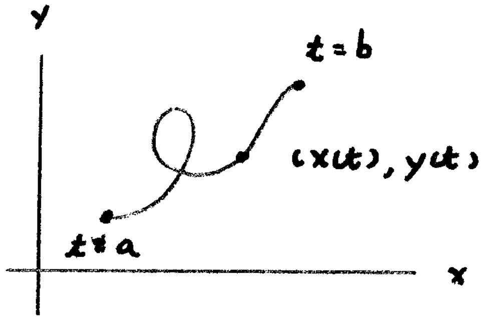
$$= \tfrac{1}{8}y^4\Big|_1^4 - \tfrac{1}{4}y^{-2}\Big|_1^4$$

$$= \tfrac{1}{8}(256-1) - \tfrac{1}{4}\left(\tfrac{1}{16} - 1\right)$$

$$= \frac{255}{8} - \tfrac{1}{4}\left(-\tfrac{15}{16}\right) = \frac{255}{8} + \frac{15}{64}$$

$$= \frac{2040}{64} + \frac{15}{64} = \frac{2055}{64}$$

SOMETIMES A CURVE IN THE XY-PLANE IS DESCRIBED PARAMETRICALLY BY GIVING THE x AND y COORDINATES OF A POINT ON THE CURVE AS FUNCTIONS OF SOME PARAMETER t (THINK OF t AS TIME AND THE CURVE AS THE PATH OF AN OBJECT MOVING IN THE PLANE).



ARGUMENTS JUST LIKE THOSE GIVEN EARLIER SHOW

$$\text{LENGTH} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

<u>EXAMPLE</u> : $(x(t), y(t)) = (a\cos t, a\sin t)$, $0 \le t \le 2\pi$

(WITH $a > 0$ A CONSTANT) DESCRIBES A CIRCLE OF RADIUS a

BECAUSE

$$\begin{aligned} (x(t))^2 + (y(t))^2 &= (a\cos t)^2 + (a\sin t)^2 \\ &= a^2\cos^2 t + a^2 \sin^2 t \\ &= a^2(\cos^2 t + \sin^2 t) \\ &= a^2 \end{aligned}$$

IT'S LENGTH IS

$$\int_0^{2\pi} \sqrt{((a\cos t)')^2 + ((a\sin t)')^2}\, dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{2\pi} a\, dt = at\Big|_0^{2\pi} = 2\pi a$$

AS EXPECTED.