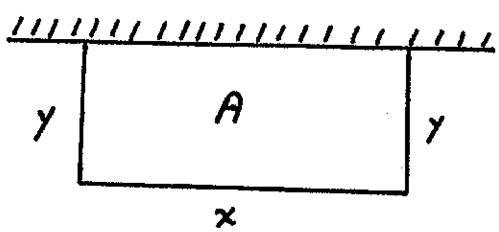
APPLIED MAXIMUM-MINIMUM PROBLEMS :

I'll TRY TO ILLUSTRATE THE GENERAL FEATURES OF THESE PROBLEMS AND SOME RULES OF THUMB FOR SOLVING THEM WITH A SIMPLE EXAMPLE. THEN WE'LL GO THROUGH A NUMBER OF MORE INVOLVED EXAMPLES.

> 100 FT OF FENCING IS AVAILABLE WITH WHICH TO ENCLOSE A RECTANGULAR GARDEN. ONE SIDE OF THE GARDEN IS TO BE ALONG THE SIDE OF A HOUSE AND THEREFORE REQUIRES NO FENCING. HOW SHOULD THE DIMENSIONS OF THE GARDEN BE CHOSEN IN ORDER TO MAXIMIZE THE ENCLOSED AREA ?

GENERAL FEATURES : CHOOSE SOMETHING (E.G., DIMENSIONS) IN ORDER TO MAXIMIZE OR MINIMIZE SOMETHING ELSE (E.G., AREA).

PROCEDURE : DRAW A PICTURE (IF POSSIBLE) AND INTRODUCE NAMES FOR THE THINGS YOU ARE TO CHOOSE AND THE THING TO BE MAXIMIZED OR MINIMIZED.



x = LENGTH OF GARDEN

y = WIDTH OF GARDEN

A = AREA OF GARDEN

WRITE THE THING TO BE MAXIMIZED/MINIMIZED IN TERMS OF THE THINGS TO BE CHOSEN :

$$A = xy$$

USE ANY CONSTRAINT IMPOSED ON THE WAY YOU CAN MAKE YOUR CHOICE (E.G., ONLY 100 FT OF FENCING AVAILABLE) TO ELIMINATE ALL BUT ONE OF THE "CHOICE" VARIABLES.

$$x + y + y = 100$$
$$x + 2y = 100$$
$$y = 50 - \tfrac{1}{2}x$$

$$A = x\left(50 - \tfrac{1}{2}x\right)$$
$$A(x) = 50x - \tfrac{1}{2}x^2$$

THE RESULT IS A FUNCTION OF ONE VARIABLE.

IF THE PROBLEM IMPOSES A RESTRICTION ON VALUES OF THE CHOICE VARIABLE, WRITE IT AS AN INTERVAL.

$$0 \le x \le 100$$

<u>NOTE</u> : IT MAY SEEM SILLY TO INCLUDE $x = 0$ AND $x = 100$ SINCE BOTH GIVE AN AREA OF 0, BUT A <u>CLOSED</u> INTERVAL IS ALWAYS PREFERABLE SINCE ABSOLUTE MAXIMA AND MINIMA ALWAYS EXIST.

NOW FIND THE MAXIMUM/MINIMUM VALUE OF THIS FUNCTION ON THIS INTERVAL.

MAXIMIZE $A(x) = 50x - \frac{1}{2}x^2$ ON $[0, 100]$.

CRITICAL POINTS : $A'(x) = 50 - x$

$x = 50$

$A(50) = 50(50) - \frac{1}{2}(50)^2$

ENDPOINTS :

$A(50) = 1250$
$A(0) = 0$
$A(100) = 0$

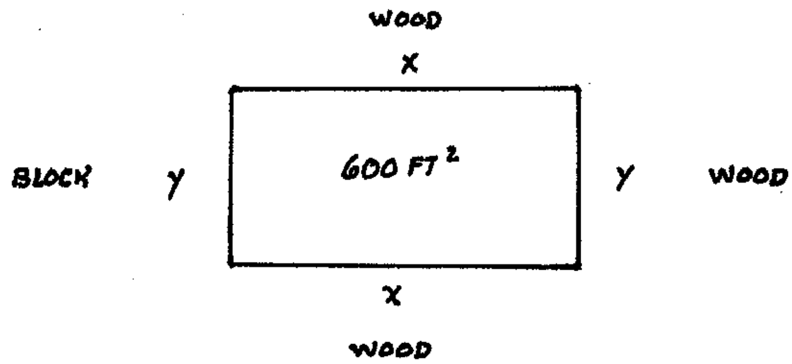
MAXIMUM OF $1250\ FT^2$ WHEN

$x = 50$ FT

$y = 50 - \frac{1}{2}(50) = 25$ FT

<u>MORE</u> <u>EXAMPLES</u> :

1. A RECTANGULAR ENCLOSURE IS TO BE CONSTRUCTED WITH AREA $600\ FT^2$. THREE SIDES ARE TO BE MADE OF WOOD (COST : \$7.00 PER FT) AND THE FOURTH SIDE IS TO BE MADE OF CEMENT BLOCK (COST : \$14.00 PER FT). WHAT DIMENSIONS WILL MINIMIZE THE COST ?



C = COST

$$C = 7x + 7x + 7y + 14y = 14x + 21y$$

CONSTRAINT: $xy = 600$

$$y = \frac{600}{x}$$

THUS,

$$C = 14x + 21\left(\frac{600}{x}\right)$$

$$C(x) = 14x + \frac{12{,}600}{x}$$

RESTRICTION: $x > 0$

MINIMIZE $C(x) = 14x + \frac{12{,}600}{x}$ ON $(0, \infty)$

NOTE THAT $\lim\limits_{x \to \infty} \left(14x + \frac{12{,}600}{x}\right) = \infty$ AND $\lim\limits_{x \to 0^+} \left(14x + \frac{12{,}600}{x}\right) = \infty$

SO THERE IS AN ABSOLUTE MINIMUM ON $(0, \infty)$ THAT MUST OCCUR AT A CRITICAL POINT.

$$C'(x) = 14 - \frac{12{,}600}{x^2}$$

DEFINED EVERYWHERE ON $(0, \infty)$. $C'(x) = 0$ WHEN

$$14 - \frac{12{,}600}{x^2} = 0$$

$$x^2 = \frac{12,600}{14} = 900$$

$$x = 30 \quad (\text{SINCE } x > 0)$$

MINIMUM COST:

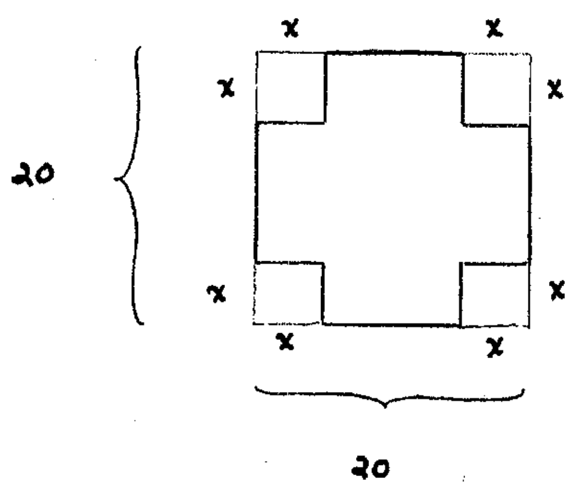
$$C(30) = 14(30) + \frac{12,600}{30} = \$840$$

OPTIMAL DIMENSIONS:

$$x = 30 \text{ FT}$$

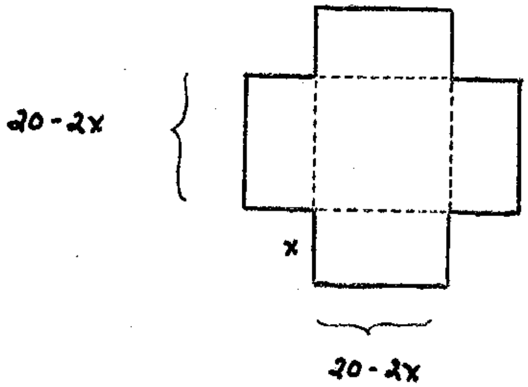
$$y = \frac{600}{30} = 20 \text{ FT}$$

2. A SQUARE SHEET OF CARDBOARD 20 IN ON A SIDE IS TO BE USED TO MAKE AN OPEN TOP BOX BY CUTTING A SMALL SQUARE FROM EACH CORNER AND THEN FOLDING THE SIDES UP. HOW LARGE A SQUARE SHOULD BE CUT FROM EACH CORNER TO MAXIMIZE THE VOLUME OF THE RESULTING BOX?



X = SIDE LENGTH OF THE SQUARE CUT OUT

V = VOLUME OF THE RESULTING BOX



FOLD FLAPS UP ALONG THE DOTTED LINES

$$V = (\text{LENGTH})(\text{WIDTH})(\text{HEIGHT})$$

$$= (20-2x)(20-2x)(x)$$

$$V(x) = 4x^3 - 80x^2 + 400x$$

$$0 \leq x \leq 10$$

$$V'(x) = 12x^2 - 160x + 400 = 4(3x^2 - 40x + 100)$$

$$= 4(3x-10)(x-10)$$

CRITICAL POINTS IN $(0 < x < 10)$: $x = \frac{10}{3}$

$$V\left(\tfrac{10}{3}\right) = \left(20 - \tfrac{20}{3}\right)\left(20 - \tfrac{20}{3}\right)\left(\tfrac{10}{3}\right)$$

$$\boxed{V\left(\tfrac{10}{3}\right) = \frac{16{,}000}{9}}$$

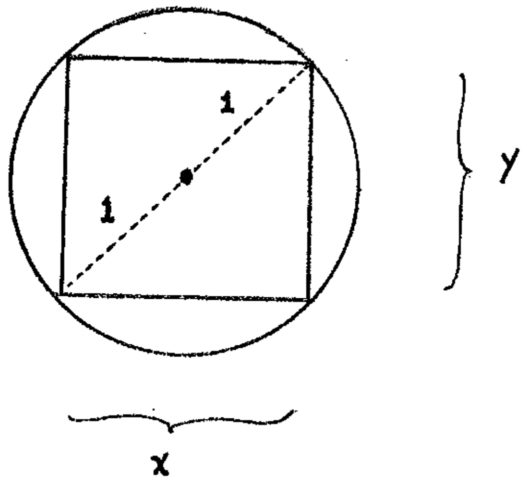
ENDPOINTS :

$$\boxed{V(0) = 0}$$

$$\boxed{V(10) = 0}$$

MAXIMUM VOLUME OF $\frac{16{,}000}{9}$ IN^3 WHEN $x = \frac{10}{3}$ IN

3. (SAWMILL PROBLEM) FIND THE RECTANGLE OF MAXIMAL AREA THAT CAN BE INSCRIBED IN A CIRCLE OF RADIUS 1.



x = LENGTH OF RECTANGLE

y = WIDTH OF RECTANGLE

A = AREA OF RECTANGLE

$$A = xy$$

PYTHAGOREAN THEOREM GIVES

$$x^2 + y^2 = 2^2$$
$$y = \sqrt{4-x^2}$$

$$A(x) = x\sqrt{4-x^2}$$

$$0 \le x \le 2$$

CRITICAL POINTS IN $(0,2)$:

$$\begin{aligned} A'(x) &= \left(x(4-x^2)^{\frac{1}{2}}\right)' = x\left(\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)\right) + (4-x^2)^{\frac{1}{2}} \\ &= -x^2(4-x^2)^{-\frac{1}{2}} + (4-x^2)^{\frac{1}{2}} \\ &= (4-x^2)^{-\frac{1}{2}}\left(-x^2 + (4-x^2)\right) \\ &= \frac{4-2x^2}{\sqrt{4-x^2}} \end{aligned}$$

$$A'(x) = \frac{4-2x^2}{\sqrt{4-x^2}}$$ IS DEFINED EVERYWHERE ON $(0,2)$ AND

EQUALS 0 ONLY WHEN

$$4-2x^2 = 0$$
$$x^2 = 2$$
$$x = \sqrt{2} \qquad (-\sqrt{2} \text{ IS NOT IN } (0,2))$$

$$A(\sqrt{2}) = \sqrt{2}\sqrt{4-(\sqrt{2})^2} = \sqrt{2}\sqrt{2}$$

$$\boxed{A(\sqrt{2}) = 2}$$

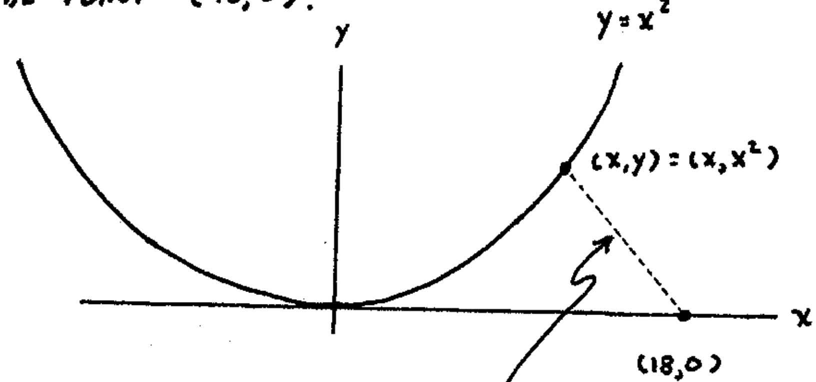
ENDPOINTS :

$A(0) = 0$
$A(2) = 0$

MAXIMUM AREA OF 2 WHEN

$$x = \sqrt{2}$$
$$y = \sqrt{4-(\sqrt{2})^2} = \sqrt{2}$$

4. FIND THE POINT ON THE GRAPH OF $y = x^2$ THAT IS CLOSEST TO THE POINT $(18,0)$.



MINIMIZE THE LENGTH L OF THE LINE JOINING $(18,0)$ WITH A POINT ON THE GRAPH

FROM THE DISTANCE FORMULA, FOR ANY $(x,y) = (x,x^2)$ ON THE GRAPH,

$$L = \sqrt{(x-18)^2 + (y-0)^2}$$

$$= \sqrt{(x-18)^2 + (x^2-0)^2}$$

$$= \sqrt{x^4 + x^2 - 36x + 324}$$

NOTE : IT WILL SIMPLIFY THE ARITHMETIC TO NOTICE THAT MINIMIZING THIS IS THE SAME AS MINIMIZING

$$f(x) = x^4 + x^2 - 36x + 324$$

SINCE $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$ THERE IS AN ABSOLUTE MINIMUM WHICH MUST OCCUR AT A CRITICAL POINT.

$$f'(x) = 4x^3 + 2x - 36$$

$$f'(x) = 0$$

$$4x^3 + 2x - 36 = 0$$

$$2x^3 + x - 18 = 0$$

ONE SOLUTION IS EASY TO FIND BY INSPECTION : $x = 2$

THUS, $2x^3 + x - 18$ HAS A FACTOR OF $x - 2$. TO FIND THE OTHER FACTOR WE DO A DIVISION :

$$
\begin{array}{r}
2x^2 + 4x + 9 \\
x-2 \overline{\smash{)}\, 2x^3 + 0x^2 + x - 18} \\
\underline{2x^3 - 4x^2} \qquad\qquad \\
4x^2 + x \qquad \\
\underline{4x^2 - 8x} \qquad \\
9x - 18 \\
\underline{9x - 18} \\
0
\end{array}
$$

THUS,

$$2x^3 + x - 18 = (x-2)(2x^2+4x+9)$$

SO

$$2x^3 + x - 18 = 0$$

$$(x-2)(2x^2+4x+9) = 0$$

$$x - 2 = 0 \qquad\qquad 2x^2+4x+9 = 0$$

$$x = 2$$

HAS ONLY COMPLEX ROOTS SINCE

$$b^2 - 4ac = 4^2 - 4(2)(9) < 0$$

CONSEQUENTLY, THERE IS ONLY ONE CRITICAL POINT, WHICH MUST GIVE THE ABSOLUTE MINIMUM.

CLOSEST POINT :

$$(x, x^2) = (2, 4)$$