Multiple Model Adaptive Regulation
Systems with Different Zero Structures

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Motivating Example

Example: Longitudinal, 4 state, Nonlinear Aircraft Dynamics

- Sum Forces and Moments in body axis:
  \[ \Sigma F_x, \Sigma F_z, \Sigma M_y \]
- Nonlinear dynamics are
  \[ \dot{x} = f(x, \kappa) + g(x, u, \kappa) \]
- \( R(v, \alpha) \) rotates to wind coordinates,
  \[ x = [v, \alpha, \theta, q]^T, \quad u = [T, \delta], \]
  \( \kappa \) is the C.G.

The Equations of Motion are

\[
R(v, \alpha)^{-1} \dot{x} = \begin{bmatrix}
W \sin \theta + D \cos \alpha + L_w \sin \alpha \\
W \cos \theta - D \sin \alpha - L_w \cos \alpha \\
q \\
M_w + \kappa l_w L_w \cos \alpha - c q
\end{bmatrix}
+ \begin{bmatrix}
T + L_t \sin \alpha_t(\delta) \\
- L_t \cos \alpha_t(\delta) \\
0 \\
-(1 - \kappa) l_t L_t \cos \alpha_t(\delta)
\end{bmatrix}
\]
Flight Safety

- Safety critical systems operate with degraded conditions & faults.
- Regulate outputs despite uncertain change in system structure
- Continuous Adaptive Control methods may fail.
- Flight Safety examples of structural change include:
  - control reversal
  - component failure
  - engine compressor rotating stall
Outline

1. Dynamics and Linear Systems
   - Freq. & Time Domain Analysis
   - Regulation
   - Lyapunov Stability

2. MMAR Design Summary
   - LMI Formulation
   - Common Quadratic Formulation

3. Design Details
   - Calculations
   - Aero Example
Problem Classification

Statics: $\Sigma F = 0$

Dynamics: $\Sigma F = ma$

Consider a mass spring damper system:

$$ma + cv + kx = u(t)$$

Analysis methods:
- Laplace Transform (Frequency Domain)
- Time Domain
Frequency Domain Analysis

- Solution Steps in Frequency Domain:
  1. Laplace Transform: $\mathcal{L}\{f(t)\} \rightarrow F(s)$
  2. Algebra with $F(s)$
  3. Inverse Laplace Transform: $\mathcal{L}^{-1}\{F(s)\} \rightarrow f(t)$

Laplace Transform of Constant (Step Function)

$L\{f(t)\} = \int_{0}^{\infty} f(t) e^{-st} dt$

$L\{1\} = \int_{0}^{\infty} 1 e^{-st} dt = \left[ -\frac{1}{s} e^{-st} \right]_{0}^{\infty} = \frac{1}{s}$

A step function has $F(s) = \frac{1}{s}$
Solution Steps in Frequency Domain:

1. Laplace Transform: $\mathcal{L}\{f(t)\} \rightarrow F(s)$
2. Algebra with $F(s)$
3. Inverse Laplace Transform: $\mathcal{L}^{-1}\{F(s)\} \rightarrow f(t)$

Laplace Transform of Constant (Step Function)

$\mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t) e^{-st} dt$

$\mathcal{L}\{1^{+}\} = \int_{0}^{\infty} 1 e^{-st} dt = -\frac{1}{s} e^{-st} \bigg|_{0}^{\infty} = -\frac{1}{s} \left[ e^{-s\infty} - e^{-s0} \right] = \frac{1}{s}$

A step function has $F(s) = \frac{1}{s}$
Frequency Domain Transfer Function:

- Given system: \( ma + cv + kx = u(t) \)
- Laplace Transform: \( (ms^2 + cs + k)X = U(s) \)
- Transfer Function Form: \( \frac{X}{U} = \frac{1}{ms^2 + cs + k} = \frac{N(s)}{D(s)} \)

where the roots of \( D(s) \) & \( N(s) \) are the poles & zeros.
Frequency Domain Analysis (cont.)

- Frequency Domain Transfer Function:
  - Given system: \( ma + cv + kx = u(t) \)
  - Laplace Transform: \( (ms^2 + cs + k)X = U(s) \)
  - Transfer Function Form: \( \frac{X}{U} = \frac{1}{ms^2 + cs + k} = \frac{N(s)}{D(s)} \)

  where the roots of \( D(s) \) & \( N(s) \) are the poles & zeros.

- Factor the pole polynomial \( D(s) \) as
  \[
  D(s) = (s + p_1)(s + p_2)\ldots(s + p_n) = 0
  \]

  The system is stable iff \( p_i \in \mathbb{C}^- \), \( i = \{1, 2, \ldots, n\} \)
An $n$’th order differential equation can be written as a system of $n$ first order differential equations:

\[ x = x_1 \]
\[ \frac{d}{dt} x = x_2 \]
\[ \vdots \]
\[ \frac{d^{n-1}}{dt^{n-1}} x = x_n \]

In our case, $x_1 = \text{position (x)}$, $x_2 = \text{velocity (v)}$. 
Time Domain Analysis (cont.)

- Write \( ma + cv + kx = u(t) \) as a system of equations:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -kx_1 - cx_2 + u(t)
\end{align*}
\]

- In matrix form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-k & -c
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u(t)
\]
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0 \\
1
\end{bmatrix} u(t)
\]

As linear time invariant system

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

where \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ldots \)
Given a linear system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &=Cx
\end{align*}
\]

**Poles** Compute \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) the set of eigenvalues of \( A \).

The system is stable iff \( \lambda_i \in \mathbb{C}^- \), \( i = \{1, 2, \ldots, n\} \)

**Zeros** Values of \( s \) where the system matrix \( \Gamma(s) \) loses rank

\[
\Gamma(s) = \begin{bmatrix}
sI - A & B \\
C & 0
\end{bmatrix}
\]

The zeros are unchanged by state feedback.
LTI Regulation with state disturbance and reference tracking:

Plant \[
\dot{x} = Ax + Bu + E\vartheta
\]

Exosystem \[
\dot{\vartheta} = Z\vartheta
\]

Regulated Outputs \[
e = Cx - F\vartheta
\]
LTI Regulation with state disturbance and reference tracking:

\[
\begin{align*}
\text{Plant} & \quad \dot{x} = Ax + Bu + E\dot{\vartheta} \\
\text{Exosystem} & \quad \dot{\vartheta} = Z\vartheta \\
\text{Regulated Outputs} & \quad e = Cx - F\vartheta
\end{align*}
\]

Regulation requires

1. Output Regulation: \( \lim_{t \to \infty} e(t) = 0 \)
2. Internal Stability: The plant and controller states are bounded
Robust Regulation of Linear Plants

- LTI Regulation with state disturbance and reference tracking:

\[
\begin{align*}
\text{Plant} & : \quad \dot{x} = A_\theta x + B_\theta u + E_\theta \nu \\
\text{Exosystem} & : \quad \dot{\nu} = Z_\theta \nu \\
\text{Regulated Outputs} & : \quad e = C_\theta x - F_\theta \nu
\end{align*}
\]

- Regulation requires
  1. Output Regulation: \( \lim_{t \to \infty} e(t) = 0 \)
  2. Internal Stability: The plant and controller states are bounded
Robust Regulation and Internal Model

Define:

\[ \phi(s) = (sI_r - Z) \]

\[ y_{ref} = \frac{1}{\phi(s)} \theta_0 \]

Has error dynamics:

\[ e = (y_{ref} - y) = \frac{D\phi}{D\phi + N} \cdot \frac{1}{\phi} \theta_0 \]
Robust Regulation and Internal Model

Define:

\[ \phi(s) = (sl_r - Z) \]

\[ y_{\text{ref}} = \frac{1}{\phi(s)} \vartheta_0 \]

Has error dynamics:

\[ e = (y_{\text{ref}} - y) = \frac{D\phi}{D\phi + N} \cdot \frac{1}{\phi} \vartheta_0 \]

The closed R.H.P. poles of exo. dynamics, \( \phi(s) \in C^+ \), cancel

The closed loop poles \( D\phi + N \) are stable by design
Define: \textbf{Singular surface} of codim 1 in parameter space exists

If \[ |\Gamma_\theta(0)| = 0, \] Where \[ \Gamma_\theta(s) = \begin{bmatrix} sl - A_\theta & B_\theta \\ C_\theta & 0 \end{bmatrix} \]

\textbf{Theorem [4]}: A single linear controller is unable to simultaneously regulate plants on either side of the singular surface.
Define: Singular surface of codim 1 in parameter space exists

If \( |\Gamma_\theta(0)| = 0 \), Where \( \Gamma_\theta(s) = \begin{bmatrix} sI - A_\theta & B_\theta \\ C_\theta & 0 \end{bmatrix} \)

Theorem [4]: A single linear controller is unable to simultaneously regulate plants on either side of the singular surface.

Example: Controller designed for plant \( p_a \) fails to stabilize plant \( p_b \):

\[ |\Gamma_\theta(0)| = 0 \]

Singular surfaces partition the family of plants \( \mathcal{P} \) into equivalence classes (c.f. Section 3, Future Work)
Local Lyapunov Stability

- **Lyapunov Function Properties**, \( V : \mathbb{R}^n \rightarrow \mathbb{R} \)
  
  1. \( V(0) = 0 \) and \( V(x) > 0 \)
  
  2. \( \frac{d}{dt} V(x(t)) \leq 0 \)

- \( V(x) \) is a scalar, e.g. \( \Sigma \) Energy

- \( V(x) = c \) are concentric, invariant sets

- Quadratic Lyapunov functions \( V(x) = x^T P x, \ P \succ 0 \), are ellipsoids

- Switch Logic uses \( V(x(t + dt)) \leq V(x(t)) \)
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Multiple Model Adaptive Regulation (MMAR)

Continuous Adaptive Control

Multiple Model Adaptive Regulation

Structure Change
Problem Statement

- Multiple Models address multiple plant structure.

Control Covering Problem: Given a range of plant parameters $\theta$,

1. Design a set of controllers $C$ such that each $p(\theta) \in \mathcal{P}$ is stabilized by at least one $C_i \in C$. The set $C$ is the control covering.

2. In general, $\mathcal{P}$ is continuous, whereas the controller set $C$ is discrete.

Switching Problem: Identify and apply a stabilizing $C_i \in C$ for an unknown plant, $p(\theta^*) \in \mathcal{P}$
Control Covering: Comparison with Prior Work

Robust via Induced Norm
- Freq. Domain v-gap [2]
- Lyapunov Functions [3]

Our Approach:
- identify structural impediments to continuous adaption
- ‘cover’ disjoint sets with multiple controllers
Write quadratic ARE as convex Linear Matrix Inequality (LMI) (Shur) [7]

\[(A_{cl} - \Theta)^T P + P(A_{cl} - \Theta) + Q < 0\]  (ARE)

\[\Leftrightarrow\]

\[
\begin{bmatrix}
(A_{cl} - \Theta)^T P + P(A_{cl} - \Theta) & I \\
I & -Q
\end{bmatrix} < 0 \]  (LMI)
Write quadratic ARE as convex Linear Matrix Inequality (LMI) (Shur) [7]

\[(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) + Q < 0 \quad (ARE)\]

\[
\begin{bmatrix}
(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) & I \\
I & -Q
\end{bmatrix} < 0 \quad (LMI)
\]

A single \(P\) for all \(\Theta_j \in \{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}\) in

\[
\begin{bmatrix}
(A_{cl} - \Theta_j)^T P + P (A_{cl} - \Theta_j) & I \\
I & -Q
\end{bmatrix} < 0
\]

verifies stability of \(C\) for all \(\theta \in Co \{\Theta_j\}\).
LMI: Quadratic Design

- Linear map $A_{\theta} \equiv A(\theta)$, etc.; $\Omega = \text{Convex Hull } \{\Theta_1, \ldots, \Theta_N\}$
- A simultaneous $P$ for AREs at vertices of $\Omega$, i.e. $A(\Theta_1)$, etc.

$$A_{\theta}^T P + PA_{\theta} - PB_{\theta} R^{-1} B_{\theta}^T P + Q < 0$$

computes both a Control and a Performance Metric.
LMI: Quadratic Design

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computes both a Control and a Performance Metric.

Control

- Control $u = Kx$ such that $\sigma(A + BK) \subset \mathbb{C}^-$
- LQR state feedback gain $K = -R^{-1} B_*^T P$
- $B_* \in \text{Co} \{B(\Theta_{ij})\}$

Performance Metric

- Use Lyapunov function $V(x(t)) = x(t)^T Px(t)$
- Rate of decay $\frac{d}{dt} V(x(t)) < \gamma(Q, R) < 0$
Design Summary: Cover & Switch

Find sufficiently many $C_i$ to cover the unknown $\theta^*$ by at least one $C_i$.

Switch Logic tests controller $C_i$ with Lyapunov Function in $P_i$ for acceptable performance.

$C_1$ “Covers” $\theta^*$

Lyapunov stability of $C_1$ for $\theta^*$
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Recall the exosystem is \( \dot{\vartheta} = Z\vartheta \).

Append the error system

\[ \dot{\eta} = Z\eta + Je \]

Form closed loop dynamics

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta}
\end{bmatrix} =
\underbrace{egin{bmatrix}
A & 0 \\
JC & Z
\end{bmatrix}}_{\bar{A}}
\begin{bmatrix}
x \\
\eta
\end{bmatrix} +
\underbrace{egin{bmatrix}
B \\
0
\end{bmatrix}}_{\bar{B}} u +
\begin{bmatrix}
E \\
-F
\end{bmatrix} \vartheta
\]

Design state feedback

\[
u = \begin{bmatrix} K_x & K_\eta \end{bmatrix} \begin{bmatrix} x \\
\eta \end{bmatrix} = K\bar{x} \quad \text{s.t.} \quad \sigma (\bar{A} + \bar{B}K) \subset \mathcal{C}^-
\]
Singular Surface & Error Augmentation

- Regulator block diagram:

- Closed Loop dynamics can be factored as

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
A_{\theta} + B_{\theta}Kx & B_{\theta}K_{\eta} \\
JC_{\theta} & Z
\end{bmatrix}
\begin{bmatrix}
x \\
\eta
\end{bmatrix}
\]

\[
\downarrow
\]

\[
\begin{bmatrix}
I & 0 \\
0 & J
\end{bmatrix}
\cdot
\begin{bmatrix}
A_{\theta} & B_{\theta} \\
C_{\theta} & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
I & 0 \\
K_{x} & K_{\eta}
\end{bmatrix}
\]

- Stability is lost when an eigenvalue of $\Gamma_{\theta}(0)$ changes sign [4]
Quadratic Subproblems

- Two design problems:
  1. Find a $K_i$ for each $C_i$
  2. Find a metric to identify a stabilizing $C_i$

- These two problems are quadratic:
  1. $K$ is the Linear Quadratic Regulator state feedback gain
  2. $P$ is the Common Quadratic Lyapunov function,
     \[ V_i = x^T P_i x \]

- One Linear Matrix Inequality obtains $K$ and $P$ in polynomial time.

- The controller set covering problem is simplified:
  - Previous methods use matrix norms
  - Individual controllers often have significant overlap.
  - Controller covering of the entire $\theta$ space is hard to verify.

- Here $\theta$ is "covered" by convex polytopes
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One Linear Matrix Inequality obtains $K$ and $P$ in polynomial time.

The controller set covering problem is simplified:
- Previous methods use matrix norms
  - Individual controllers often have significant overlap.
  - Controller covering of the entire $\theta$ space is hard to verify.
- Here $\theta$ is “covered” by convex polytopes
Consider the system

\[
\dot{x} = Ax + Bu \\
z = Cx + Du
\]

Minimize the output energy

\[
\int_0^\infty z^T z \, dt
\]

By solving the riccati equation

\[
A^T P + PA + C^T C - PB \left( D^T D \right)^{-1} B^T P = 0
\]

The two subproblem solutions are thus

1. The state feedback gain is \( K = - \left( D^T D \right)^{-1} B^T P \)
2. A switch metric is Quadratic Lyapunov function \( V = x^T Px \)
Consider the system
\[ \dot{x} = A_j x + B_j u \]
\[ z = Cx + Du \]

Minimize the output energy
\[ \int_0^\infty z^T z \, dt \]

By solving the riccati equation
\[ A_j^T P_i + P_i A_j + C^T C - P_i B_j (D^T D)^{-1} B_j^T P_i < 0 \]

The two subproblem solutions are thus

1. The state feedback gain is \( K_i = - (D^T D)^{-1} B_*^T P_i \)

2. A switch metric is Common Quadratic Lyapunov function \( V_i = x^T P_i x \)
Linear Matrix Inequality (LMI)

- Iteratively divide $\mathcal{P}$ into subfamilies, one $C_i$ per subfamily
- 3 subfamilies imply 3 polytopic LMIs, $i \in \{1, 2, 3\}$
- Let $\theta_{ij}$ refer to subfamily $i$, vertex $j \in \{1, \ldots, v_i\}$
- Polytopic LMI is $\text{blkdiag } v_i$ LMIs in $\bar{A}_{ij} = \begin{bmatrix} A_{\theta_{ij}} & 0 \\ J_i C_{\theta_{ij}} & Z \end{bmatrix}$, $\bar{B}_{ij} = \begin{bmatrix} B_{\theta_{ij}} \\ 0 \end{bmatrix}$

- $\mathcal{P}$ has 3 subfamilies, $i = \{1..3\}$
- $C_1$ has 4 vertices, $j = \{1..4\}$

$$LMI_{C_1} = \begin{bmatrix} LMI_{\theta_{11}} \\ \vdots \\ LMI_{\theta_{14}} \end{bmatrix}$$
Switching

- Switch logic uses a Lyapunov metric
  - By design at least one $C_i$ stabilizes $\rho(\theta^*) \in \mathcal{P}$
  - Each $C_i$ has a bound on the rate of decay of $V_i$

\[ V_i(\tau + dt) > \gamma V_i(\tau) \text{ for } dt > 0, \ 0 < \gamma \leq 1, \]

Then reject $C_i$ & remove $i$ from set of feasible indices,

Switch to next feasible index [8]

- Closed loop trial and error locates a stabilizing controller.
Find sufficiently many $C_i$ to cover the unknown $\theta^*$ by at least one $C_i$

$C_i$, $P_i$, and Switching

- $\dot{x} = f(\theta^*, C_i)$
- $V_i = x^T P_i x$
- By design, $\frac{d}{dt} V_i < 0$ for some $C_i$
- If $V_i(\tau + dt) > \gamma V_i(\tau)$, Switch off $C_i$

Example Lyapunov function for $\theta^*$ & $C_1, P_1$
Example: Aircraft Center of Gravity change

- Abstract the longitudinal aircraft dynamics of [4]:

\[
A_\theta = \begin{bmatrix}
\theta_1 & 0 & 1 \\
0 & 0 & 1 \\
\theta_2 & 0 & \theta_3
\end{bmatrix}, \quad B_\theta = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad C_\theta = \begin{bmatrix}
1 & 1 & 0
\end{bmatrix}, \quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

where \( \{x_1, x_2, x_3\} \sim \{\alpha, \theta, q\} \). The relative degree of \( \alpha \) and the zero structure from \( \theta, q \) are preserved.

- Transfer Function:

\[
C_\theta (sI - A_\theta)^{-1} B_\theta = \frac{s^2 - s\theta_3 + \theta_2}{D(s, \theta_1, \theta_3)}
\]

- At \( \theta_2 = 0 \) the system is structurally unstable with
  - a transmission zero at \( s = 0 \),
  - a pole at the origin,
  - \((A_\theta, B_\theta)\) uncontrollable.
Aircraft Example: Subfamilies

- In between the subfamilies, $\theta_2 \approx 0$, controllability and regulation are compromised.

- A single, structurally stable controller fails to regulate plants from both subfamilies.

- Solve two LMIs for the SFBK controller and Lyapunov matrix:
  - $\theta_2 \in [-4, -1] = \text{Top Branch: } K_T, P_T$
  - $\theta_2 \in [1, 4] = \text{Bottom Branch: } K_B, P_B$
Aircraft Example: Control Design

▶ Design weights $Q_z = \text{diag}([1, 1, 25, 25]) / 10$ and $R_z = 1$, Obtain:

$$K_B = [-3.5, -8.1, -2.6, -2.7] \quad P_B = \begin{bmatrix}
3.5 & 8.1 & 2.6 & 2.7 \\
8.1 & 74. & 18. & 10. \\
2.6 & 18. & 6.3 & 1.8 \\
2.7 & 10. & 1.8 & 16. \\
\end{bmatrix}$$

$$K_T = [-18, 174, 39, 25] \quad P_T = \begin{bmatrix}
-174. & 8224. & 1736. & 1230. \\
-39. & 1736. & 389. & 259. \\
-25. & 1230. & 259. & 202. \\
\end{bmatrix}$$

▶ Solve $\dot{x} = f(x_f, \theta, K, \vartheta) = 0$ with state, gain, and set point:

$$x = [x_1, x_2, x_3, \eta]^T, \quad K = [k_1, k_2, k_3, k_4], \quad \vartheta_r,$$

$$x_f = \begin{bmatrix}
-\frac{\vartheta}{\theta_2} & \vartheta \left(1 + \frac{1}{\theta_2}\right) & 0 & \frac{\vartheta(\theta_1 + k_1 - k_2(1 + \theta_2))}{k_4 \theta_2} \\
\end{bmatrix}^T$$
Aircraft Example: Simulation, Performance

Initial Conditions

\[
\begin{bmatrix}
  x_1(0) & x_2(0) & x_3(0) & \eta(0)
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
\]

\[\dot{\vartheta}(0) = -0.5\]

\[\sigma(0) = T\]

\[Z = 0, \quad F = -1, \quad E^T = [0 \ 0 \ 1], \quad \theta^* = (\theta_1^*, \theta_2^*, \theta_3^*) = (-4, 1, -4)\]

\[\frac{d}{dt} (x - x_f)^T P_i (x - x_f)\]
Aircraft Example: Simulation, Stability

Error Convergence

State Performance

Internal Stability
Consider a nonlinear, affine system with regulated output $y$

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

In feedback linearized normal form ($z_1 = y$, $z_2 = \dot{y}$, ...):

$$\dot{z} = Az + E [\alpha(x) + \rho(x)u]$$
$$\dot{\xi} = F(\xi, z, u)$$

Relate the singular surface of linear regulation to changes in $\rho$ and the stability of zero dynamics $\xi$. 


Questions or Comments?
<table>
<thead>
<tr>
<th>Experience</th>
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| **Ag** | ![Image of agricultural equipment]  
| **Boeing** | ![Image of Boeing aircraft]  
| **GDRS** | ![Image of GDRS]  

Previous Work: State Space Covering [1]

- Given $A_{cl} = A + BK$ such that $A_{cl}^T P + PA_{cl} + I < 0$
- Additive uncertainty: $A + BK - \Theta \equiv A_{cl} - \Theta$
- Perturbed dynamics are Lyapunov stable if:

$$(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) + I < 0$$
Previous Work: State Space Covering [1]

- Given $A_{cl} = A + BK$ such that $A_{cl}^T P + PA_{cl} + I < 0$
- Additive uncertainty: $A + BK - \Theta = A_{cl} - \Theta$
- Perturbed dynamics are Lyapunov stable if:

$$
(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) + I < 0
$$

$$
A_{cl}^T P + PA_{cl} < -I + \Theta^T P + P \Theta
$$

$$
A_{cl}^T P + PA_{cl} < -I + 2 \|P\|_2 \|\Theta\|_2
$$

- Obtain bound on $\|\Theta\|_2$: $2 \|P\|_2 \|\Theta\|_2 < 1$
- For best case ($K_B$), $\frac{\delta \theta_1}{\theta_1} < 1\%$, $\frac{\delta \theta_2}{\theta_2} < 0.2\%$
- Therefore, need $C_i$, $i \in \{1, \ldots, N\}$, $N > 500$
Results

- Quadratic $\rightarrow$ LMI formulation:
  - Accommodates linear parametric uncertainty
  - Efficient computation and covering

- Show flexibility in choice of $B_* \in Co \{ B_{ij} \}$

- Restriction to Error Augmentation type regulator synthesis

- Lyapunov Level Set switch logic for Regulation
Consider an SISO, LTI system as

\[ p(\theta) = C_\theta (sI - A_\theta)^{-1} B_\theta = \frac{k(\theta) [s + z_1(\theta)] \cdots}{[s + p_1(\theta)] [s + p_2(\theta)] \cdots} = \frac{N(s, \theta)}{D(s, \theta)} \]

Partitions: sets of \( \theta \) give \( p(\theta) \) unique (Kronecker) zero structure
Consider an SISO, LTI system as

\[ p(\theta) = C_\theta (sI - A_\theta)^{-1} B_\theta = \frac{k(\theta) [s + z_1(\theta)] \ldots}{[s + p_1(\theta)] [s + p_2(\theta)] \ldots} = \frac{N(s, \theta)}{D(s, \theta)} \]

Partitions: sets of \( \theta \) give \( p(\theta) \) unique (Kronecker) zero structure

Two broad categories of zero structure change:

<table>
<thead>
<tr>
<th>Zero/Pole/Gain</th>
<th>Transfer Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmission Zero at the Origin:</td>
<td>( z_1(\theta) = 0 )</td>
</tr>
<tr>
<td>Input/Output Degeneracy:</td>
<td>( k(\theta) = 0 )</td>
</tr>
</tbody>
</table>
Two Regulation Methods [KK78]

- **Error Augmentation**

  - State estimation optional
  - Design state feedback gain $K$

- **Disturbance Estimation**

  - Estimate states $[\mathbf{x} \, \mathbf{v}]$
  - Design $K$, $L$, subject to $\sigma (\mathbf{Z}) \subset \sigma (\bar{\mathbf{A}} + L\bar{\mathbf{C}} + \bar{\mathbf{B}}K)$
Future Work

1. Final State: Need to know $x_f$ to compute $\frac{\Delta V(t)}{\Delta t}$ where

$$V(t) = (x(t) - x_f)^T P (x(t) - x_f)$$

2. Switching Strategy: If the set of controllers is large, “trial and error” switch logic may perform poorly.

3. MMAR for a physically meaningful Aero Example
Future Work (Step 2.)

2. Switching Strategy: If $|C| \gg 1$, performance deteriorates

A Lyapunov switch logic accommodates System Identification to accurately select the next “in-the-loop” control.
Closed loop SISO system with plant, controller, and disturbance states,

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta} \\
\dot{\omega}
\end{bmatrix} =
\begin{bmatrix}
A_\theta + B_\theta K_1 & B_\theta K_2 & E \\
JC_\theta & Z & -JF \\
0 & 0 & Z
\end{bmatrix}
\begin{bmatrix}
x \\
\eta \\
\omega
\end{bmatrix}
\]

Consider the eigenvectors \((A_{cl} - \lambda I) v = 0\) for \(\lambda_z \in \text{spectrum } Z\)

\((Z, J)\) is controllable, so \(J\) has full column rank and right inverse \(JJ^\dagger = I\)

Row 2 of \(A_{cl}\) is \(J \begin{bmatrix} C_\theta & J^\dagger (Z - \lambda_z) & -F \end{bmatrix}\)

Eigenvectors \(v_z \subset \ker \begin{bmatrix} C_\theta & 0 & -F \end{bmatrix}\), the null space of \(e = C_\theta x - F\omega\)
Proposition on $B_\star \in Co \{ B_{ij} \}$

- **Proposition**: If $P$ exists to satisfy the $j$ inequalities:

$$A^T P + PA - PB_j R^{-1} B_j^T P - PB_j R^{-1} B_j^T P < -Q, \quad \forall j \quad (1)$$

Then any $B_\star \in Co \{ B_{ij} \}$ can be chosen for the state feedback gain

$$K = -R^{-1} B_\star^T P$$

to satisfy the CQLF

$$A^T P + PA + K^T B_j^T P + PB_j K < -Q, \quad \forall j$$
Proposition on $B_* \in Co \{B_{ij}\}$

- **Proposition:** If $P$ exists to satisfy the $j$ inequalities:

$$A^T P + PA - PB_j R^{-1} B_j^T P - PB_j R^{-1} B_j^T P < -Q, \quad \forall j \quad (1)$$

Then any $B_* \in Co \{B_{ij}\}$ can be chosen for the state feedback gain $K = -R^{-1} B_*^T P$ to satisfy the CQLF

$$A^T P + PA + K^T B_j^T P + PB_j K < -Q, \quad \forall j$$

- For the scalar case (1) is

$$2ap + q < 2p^2 b_* b_j / r$$

which holds for $b_* \in [b_{min}, b_{max}]$. 
Proof of $B_* \in \text{Co}\{B_{ij}\}$

- Factor $(\Delta_j + I) B_j = B_*$ (Fix $\Omega_i$)

- Controllability: $\begin{bmatrix} I & A & A^2 & \ldots \end{bmatrix} (\Delta_j + I)^{-1} B_*$ implies $\Delta_j < I$

- Given: $A^T P + PA + K^T B_j^T P + PB_j K < -Q$, $\forall j$, $K = -R^{-1} B_j^T P$

- Then set $X = P^{-1}$, substitute $B_* = (\Delta_j + I) B_j$:

$$B_* R^{-1} B_j^T + B_j R^{-1} B_* > XA^T + AX + XQX$$

$$\left(\Delta_j + I\right) B_j R^{-1} B_j^T + B_j R^{-1} B_j^T \left(I + \Delta_j^T\right) >$$

$$B_j R^{-1} B_j^T + B_j R^{-1} B_j^T + \Delta_j B_j R^{-1} B_j^T + B_j R^{-1} B_j^T \Delta_j^T >$$

- The identity $A \succeq B$, $C \succeq D$, $A + C \succeq B + D$ implies

$$B_j R^{-1} B_j^T + B_j R^{-1} B_j^T > \Delta_j B_j R^{-1} B_j^T + B_j R^{-1} B_j^T \Delta_j^T$$
Closed loop design equation is

\[ A_d = \begin{bmatrix} A_\theta & B_\theta \bar{K}_i \\ -L_i C_\theta & \bar{A}_o + L_i \bar{C}_o + \bar{B}_o \bar{K}_i \end{bmatrix} \]

Yields a (non-convex) Bilinear Matrix Inequality

Were the observer an exact copy of the true plant dynamics, i.e., \( A_o = A_\theta \), the separation principle would hold and \( A_d \) could be transformed to an upper triangular form. Then solve for \( K_i \) in \( A_o + B_o K_i \) and again for the dual problem of \( L_i \) in \( \bar{A}_o^T + \bar{C}_o^T L_i^T \). In this idealized case a CQLF is certain for \( A_d \) in upper triangular form.
Normalized Uncertainty:

\[ T (P, C) = \left\| \begin{bmatrix} I \\ C \end{bmatrix} \cdot (I - PC)^{-1} \cdot \begin{bmatrix} P & I \end{bmatrix} \right\|_\infty \]

\( T (P, C) \cdot \delta_v (P, P_\theta) < 1 \Rightarrow \text{Stability} \)

Moreover, for unstructured uncertainty

\[ T (P, C) \cdot \delta_v (P, P_\theta) \geq 1 \Rightarrow \text{Instability} \]
v-Gap Metric applied to “Aero” example

- Compensator includes Internal Model Dynamics. Plant includes SFBK:

- Fix $\theta_1, \theta_3$. Vary $\theta_2$. Recall $T(P, C) \cdot \delta_v(P, P_{z_1}) < 1 \Rightarrow$ Stability

- Wide variation of v-Gap metric conservatism vs. LMI for “aero” model
A Strict Equivalence transformation consists of 2 operations:

1. Similarity transformation induces the familiar Controllable and Observable decomposition

\[ \mathcal{H}_{co} \oplus \mathcal{H}_{\bar{co}} \oplus \mathcal{H}_{\bar{co}} \oplus \mathcal{H}_{\bar{co}} \]

2. State feedback, Output Injection, and Scaling induce the decomposition:

\[ \mathcal{H}_{\infty} \oplus \mathcal{H}_f \oplus \mathcal{H}_\eta \oplus \mathcal{H}_\varepsilon \]

where

- ‘\( \infty \)’ are infinite divisors and describe the relative degree
- ‘\( f \)’ are finite divisors, i.e., invariant zeros
- \( \eta \) & \( \varepsilon \) are row and column indices and describe singular pencils
Feedback and Scaling are implemented in the block diagram.

Strict Equivalence is not related to stability.

Relative Degree and Transmission Zeros are preserved.
A Strict Equivalence transformation consists of 2 operations:

1. Similarity transformations of state:

\[
\begin{align*}
\bar{A} &= T^{-1}AT \\
\bar{B} &= T^{-1}B \\
\bar{C} &= CT \\
\bar{D} &= D
\end{align*}
\]

2. State feedback, output injection, and scaling ($D = 0$):

\[
\begin{align*}
\hat{A} &= \bar{A} + L\bar{C} + BF \\
\hat{B} &= \bar{B}W \\
\hat{C} &= V\bar{C} \\
\hat{D} &= 0
\end{align*}
\]
Strict Equivalence: Invariants

- Strict Equivalence is categorized by Kronecker invariants:

<table>
<thead>
<tr>
<th>Regular Pencil</th>
<th>Singular Pencil</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite divisors, $f$</td>
<td>column indices, $\varepsilon$</td>
</tr>
<tr>
<td>infinite divisors, $\infty$</td>
<td>row indices, $\eta$</td>
</tr>
</tbody>
</table>

- Regulator Design Structure differs from Strict Equivalence (S.E.):
  1. S.E. Input/Output scaling abstracts needed structure
  2. S.E. Singular pencil structure is not needed for control
Define a plant family with the form and parameter space:

\[ P = \left\{ \frac{\theta_1}{s + \theta_2} \right\} \text{ such that } \theta_1, \theta_2 \in \mathbb{R} \]

Examine the structure at \( P_a, P_b, P_c \in P \)

- \( \theta_1 \) can be less than, equal to, or greater than zero
- \( \theta_2 \) is greater than zero
Define a plant family with the form and parameter space:

\[ \mathcal{P} = \left\{ \frac{\theta_1}{s + \theta_2} \right\} \text{ such that } \theta_1, \theta_2 \in \mathbb{R} \]

Examine the structure at \( P_a, P_b, P_c \in \mathcal{P} \):

- \( \theta_1 \) can be less than, equal to, or greater than zero
- \( \theta_2 \) is greater than zero

Structure from Strict Equivalence:

\[ \left\{ \frac{1}{s}, \frac{0}{s} \right\} \]

Structure of Regulation Sub-families:

\[ \left\{ \frac{1}{s}, \frac{-1}{s} \right\} \]