Multiple Model Adaptive Regulation
Systems with Different Zero Structures

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January 15, 2014
Introduction: Hybrid Adaptive Regulation

- Safety critical systems operate with degraded conditions & faults.
- Regulate outputs despite uncertain change in system structure
- Continuous Adaptive Control methods may fail.
- Flight Safety examples of structural change include:
  - control reversal
  - component failure
  - engine compressor rotating stall
Multiple Model Adaptive Regulation (MMAR)

Continuous Adaptive Control

Multiple Model Adaptive Regulation

Structure Change
Problem Statement

- **Multiple Models** address multiple plant structure.

- **Control Covering Problem:** Given a range of plant parameters $\theta$,
  
  1. Design a set of controllers $C$ such that each $p(\theta) \in \mathcal{P}$ is stabilized by at least one $C_i \in C$. The set $C$ is the control covering.
  
  2. In general, $\mathcal{P}$ is continuous, whereas the controller set $C$ is discrete.

- **Switching Problem:** Identify and apply a stabilizing $C_i \in C$ for an unknown plant, $p(\theta^*) \in \mathcal{P}$
1. Introduction
   - Aero Example
   - Old Method: Controller Robustness
   - New Method: Convexity

2. Design Details
   - Regulation Review
   - Multi-Model Adaptive Regulation

3. Aero Example
   - Future Work
Control Covering: Comparison with Prior Work

Robust via Induced Norm
- Freq. Domain v-gap [2]
- Lyapunov Functions [3]

Our Approach:
- identify structural impediments to continuous adaption
- ‘cover’ disjoint sets with multiple controllers

\[ |\Gamma_{\theta}(0)| = 0 \]
Motivating Example

Example: Longitudinal, 4 state, Nonlinear Aircraft Dynamics

- Sum Forces and Moments in body axis:
  \[ \Sigma F_x, \Sigma F_z, \Sigma M_y \]

- Nonlinear dynamics are
  \[ \dot{x} = f(x, \kappa) + g(x, u, \kappa) \]

- \( R(v, \alpha) \) rotates to wind coordinates,
  \( x = [v, \alpha, \theta, q]^T, u = [T, \delta], \)
  \( \kappa \) is the C.G.

The Equations of Motion are

\[
R(v, \alpha)^{-1} \dot{x} = \begin{bmatrix}
W \sin \theta + D \cos \alpha + L_w \sin \alpha \\
W \cos \theta - D \sin \alpha - L_w \cos \alpha \\
q \\
M_w + \kappa I_w L_w \cos \alpha - c q
\end{bmatrix} + \begin{bmatrix}
T + L_t \sin \alpha_t (\delta) \\
- L_t \cos \alpha_t (\delta) \\
0 \\
-(1 - \kappa) I_t L_t \cos \alpha_t (\delta)
\end{bmatrix}
\]
Example: Aircraft Center of Gravity change

Abstract the longitudinal aircraft dynamics:

\[
A_\theta = \begin{bmatrix}
\theta_1 & 0 & 1 \\
0 & 0 & 1 \\
\theta_2 & 0 & \theta_3
\end{bmatrix}, \quad B_\theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_\theta = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

where \( \{x_1, x_2, x_3\} \sim \{\alpha, \theta, q\} \). The system is relative degree one and has two zeros. The parameter dependent zero structure is preserved.
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0 \\
0
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x_2 \\
x_3
\end{bmatrix}
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<table>
<thead>
<tr>
<th>Trim Condition</th>
</tr>
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<tbody>
<tr>
<td>Straight &amp; Level</td>
</tr>
<tr>
<td>High Angle of Attack</td>
</tr>
</tbody>
</table>

\[|\Gamma_\theta(0)| > 0\]
\[|\Gamma_\theta(0)| < 0\]
Aircraft Example: Parameter Space

- “AoA” & “Pitch” Damping variation:
  \[-5 < \theta_1 < -4, \quad -5 < \theta_3 < -4\]

- \(\theta_2\), or Pitch “Stiffness”, has large variation:
  - \(\theta_2 \in [-4, -1]\), is the Top subfamily
  - \(\theta_2 \in [1, 4]\), is the Bottom subfamily
Previous Work: State Space Covering [1]

- Given $A_{cl} = A + BK$ such that $A_{cl}^T P + PA_{cl} + I < 0$
- Additive uncertainty: $A + BK - \Theta \equiv A_{cl} - \Theta$
- Perturbed dynamics are Lyapunov stable if:

\[(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) + I < 0\]
Previous Work: State Space Covering [1]

- Given $A_{cl} = A + BK$ such that $A_{cl}^T P + PA_{cl} + I < 0$
- Additive uncertainty: $A + BK - \Theta = A_{cl} - \Theta$
- Perturbed dynamics are Lyapunov stable if:

\[ (A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) + I < 0 \]
\[ A_{cl}^T P + PA_{cl} < -I + \Theta^T P + P\Theta \]
\[ A_{cl}^T P + PA_{cl} < -I + 2 \|P\|_2 \|\Theta\|_2 \]

- Obtain bound on $\|\Theta\|_2$: $2 \|P\|_2 \|\Theta\|_2 < 1$
- For best case ($K_B$), $\frac{\delta\theta_1}{\theta_1} < 1\%$, $\frac{\delta\theta_2}{\theta_2} < 0.2\%$
- Therefore, need $C_i$, $i \in \{1, \ldots, N\}$, $N > 500$
Write quadratic ARE as convex Linear Matrix Inequality (LMI) (Shur) [7]

\[
(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) + Q < 0 \quad (ARE)
\]

\[
\begin{bmatrix}
(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) & I \\
I & -Q
\end{bmatrix} < 0 \quad (LMI)
\]
LMI: Convexity & Covering

> Write quadratic ARE as convex Linear Matrix Inequality (LMI) (Shur) [7]

\[
(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) + Q < 0 \quad \text{(ARE)}
\]

\[

\begin{bmatrix}
(A_{cl} - \Theta)^T P + P (A_{cl} - \Theta) & I \\
I & -Q
\end{bmatrix} < 0 \quad \text{(LMI)}
\]

> A single \( P \) for all \( \Theta_j \in \{ \Theta_1, \Theta_2, \Theta_3, \Theta_4 \} \) in

\[

\begin{bmatrix}
(A_{cl} - \Theta_j)^T P + P (A_{cl} - \Theta_j) & I \\
I & -Q
\end{bmatrix} < 0
\]

verifies stability of \( C \) for all \( \theta \in Co \{ \Theta_j \} \).
LMI: Quadratic Design

- Linear map $A_\theta \equiv A(\theta)$, etc.; $\Omega = \text{Convex Hull}\{\Theta_1, \ldots, \Theta_N\}$

- A simultaneous $P$ for AREs at vertices of $\Omega$, i.e. $A(\Theta_1)$, etc

$$A_\theta^TP + PA_\theta - PB_\theta R^{-1}B_\theta^TP + Q < 0$$

computes both a Control and a Performance Metric.
LMI: Quadratic Design

- Linear map $A_\theta \equiv A(\theta)$, etc.; $\Omega = \text{Convex Hull } \{\Theta_1, \ldots, \Theta_N\}$
- A simultaneous $P$ for AREs at vertices of $\Omega$, i.e. $A(\Theta_1)$, etc

$$A_\theta^T P + PA_\theta - PB_\theta R^{-1} B_\theta^T P + Q < 0$$

computes both a Control and a Performance Metric.

Control

- Controller $u = Kx$
- LQR state feedback gain $K = - R^{-1} B_\star^T P$
- $B_\star \in \text{Co } \{B(\Theta_{ij})\}$

Performance Metric

- Use Lyapunov function $V(x(t)) = x(t)^T P x(t)$
- Rate of decay $\frac{d}{dt} V(x(t)) < \gamma (Q, R) < 0$
Design Summary: Cover & Switch

Find sufficiently many $C_i$ to cover the unknown $\theta^*$ by at least one $C_i$.

Switch Logic tests controller $C_i$ with Lyapunov Function in $P_i$ for acceptable performance.

$C_1$ "Covers" $\theta^*$

Lyapunov stability of $C_1$ for $\theta^*$
Results

- Quadratic $\rightarrow$ LMI formulation:
  - Accommodates linear parametric uncertainty
  - Efficient computation and covering
- Show flexibility in choice of $B_* \in Co \{B_{ij}\}$
- Restriction to Error Augmentation type regulator synthesis
- Lyapunov Level Set switch logic for Regulation
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LTI Regulation with state disturbance and reference tracking:

Plant  \[ \dot{x} = A_\theta x + B_\theta u + E\dot{\vartheta} \]

Exosystem  \[ \dot{\vartheta} = Z\vartheta \]

Regulated Outputs  \[ e = C_\theta x - F\vartheta \]
LTI Regulation with state disturbance and reference tracking:

\[
\begin{align*}
\text{Plant} & : \quad \dot{x} = A_\theta x + B_\theta u + E\vartheta \\
\text{Exosystem} & : \quad \dot{\vartheta} = Z\vartheta \\
\text{Regulated Outputs} & : \quad e = C_\theta x - F\vartheta
\end{align*}
\]

Regulation requires

1. Output Regulation: \( \lim_{t \to \infty} e(t) = 0 \)
2. Internal Stability: The plant and controller states are bounded
Robust Regulation and Internal Model

Define:

\[ \phi(s) = (sl_r - Z) \]

\[ y_{ref} = \frac{1}{\phi(s)} \vartheta_0 \]

Has error dynamics:

\[ e = (y_{ref} - y) = \frac{D\phi}{D\phi + N} \cdot \frac{1}{\phi} \vartheta_0 \]
Robust Regulation and Internal Model

Define:

\[ \phi(s) = (sl_r - Z) \]

Ref. Signal

\[ y_{ref} = \frac{1}{\phi(s)} \vartheta_0 \]

Has error dynamics:

\[ e = (y_{ref} - y) = \frac{D\phi}{D\phi + N} \cdot \frac{1}{\phi} \vartheta_0 \]

The closed R.H.P. poles of exo. dynamics, \( \phi(s) \in C^+ \), cancel

The closed loop poles \( D\phi + N \) are stable by design
Consider an SISO, LTI system as

\[
p(\theta) = C_\theta (sI - A_\theta)^{-1} B_\theta = \frac{k(\theta) [s + z_1(\theta)] \ldots}{[s + p_1(\theta)] [s + p_2(\theta)] \ldots} = \frac{N(s, \theta)}{D(s, \theta)}
\]

Partitions: sets of \( \theta \) give \( p(\theta) \) unique (Kronecker) zero structure
Consider an SISO, LTI system as

\[ p(\theta) = C_\theta (sI - A_\theta)^{-1} B_\theta = \frac{k(\theta) [s + z_1(\theta)] \ldots}{[s + p_1(\theta)] [s + p_2(\theta)] \ldots} = \frac{N(s, \theta)}{D(s, \theta)} \]

- Partitions: sets of \( \theta \) give \( p(\theta) \) unique (Kronecker) zero structure
- Two broad categories of zero structure change:

<table>
<thead>
<tr>
<th>Transmission Zero at the Origin:</th>
<th>Zero/Pole/Gain</th>
<th>Transfer Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_1(\theta) = 0 )</td>
<td>( N(0, \theta) = 0 )</td>
<td>( \forall \theta )</td>
</tr>
</tbody>
</table>

| Input/Output Degeneracy:        | \( k(\theta) = 0 \) | \( N(s, \theta) = 0 \) | \( \forall s \) |
Define: **Singular surface** of codim 1 in parameter space exists

\[
\text{If } |\Gamma_\theta(0)| = 0, \quad \text{Where } \Gamma_\theta(s) = \begin{bmatrix} sl - A_\theta & B_\theta \\ C_\theta & 0 \end{bmatrix}
\]

Theorem [4]: A single linear controller is unable to simultaneously regulate plants on either side of the singular surface.
Define: Singul ar surface of codim 1 in parameter space exists

$$|\Gamma_\theta(0)| = 0,$$
where

$$\Gamma_\theta(s) = \begin{bmatrix} sI & A_\theta & B_\theta \\ C_\theta & 0 \end{bmatrix}$$

Theorem [4]: A single linear controller is unable to simultaneously regulate plants on either side of the singular surface.

Example: Controller designed for plant $p_a$ fails to stabilize plant $p_b$:

$$|\Gamma_\theta(0)| = 0$$

Singular surfaces partition the family of plants $\mathcal{P}$ into equivalence classes (c.f. Section 3, Future Work)
Two Regulation Methods [KK78]

- **Error Augmentation**
  
  - State estimation optional
  - Design state feedback gain \( K \)

- **Disturbance Estimation**
  
  - Estimate states \([x \varphi]\)
  - Design \( K, L \), subject to \( \sigma(Z) \subset \sigma(\bar{A} + L\bar{C} + \bar{B}K) \)
Recall the exosystem is \( \dot{\vartheta} = Z\vartheta \)

Append the error system

\( \dot{\eta} = Z\eta + Je \)

Form closed loop dynamics

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
    A & 0 \\
    JC & Z
\end{bmatrix}
\begin{bmatrix}
    x \\
    \eta
\end{bmatrix} +
\begin{bmatrix}
    B \\
    0
\end{bmatrix}
\begin{bmatrix}
    u \\
    v
\end{bmatrix}
\begin{bmatrix}
    E \\
    -F
\end{bmatrix}
\]

Design state feedback

\[
u = \begin{bmatrix} K_x & K_\eta \end{bmatrix} \begin{bmatrix} x \\
\eta \end{bmatrix} = K\bar{x} \quad \text{s.t.} \quad \sigma (\bar{A} + \bar{B}K) \subset C^{-}
\]
Singular Surface & Error Augmentation

- Regulator block diagram:

- Closed Loop dynamics can be factored as

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta}
\end{bmatrix} = \begin{bmatrix}
A_\theta + B_\theta K_x & B_\theta K_\eta \\
JC_\theta & Z
\end{bmatrix}
\begin{bmatrix}
x \\
\eta
\end{bmatrix}
\]

\[
\downarrow
\]

\[
\begin{bmatrix}
I & 0 \\
0 & J
\end{bmatrix}
\cdot
\begin{bmatrix}
A_\theta & B_\theta \\
C_\theta & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
I & 0 \\
K_x & K_\eta
\end{bmatrix}
\]

- Stability is lost when an eigenvalue of $\Gamma_\theta(0)$ changes sign [4]
Quadratic Subproblems

- Two design problems:
  1. Find a $K_i$ for each $C_i$
  2. Find a metric to identify a stabilizing $C_i$

- These two problems are quadratic:
  1. $K_i$ is the Linear Quadratic Regulator state feedback gain
  2. $P$ is the Common Quadratic Lyapunov function,
    \[ V_i = x^TP_i x \]

- One Linear Matrix Inequality obtains $K$ and $P$ in polynomial time.

- The controller set covering problem is simplified:
  - Previous methods use matrix norms
  - Individual controllers often have significant overlap.
  - Controller covering of the entire $\theta$ space is hard to verify.

- Here $\theta$ is "covered" by convex polytopes
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    - Controller covering of the entire $\theta$ space is hard to verify.
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Quadratic Controller Design

Consider the system
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
z &= Cx + Du
\end{align*}
\]

Minimize the output energy
\[
\int_0^\infty z^T z \, dt
\]

By solving the riccati equation
\[
A^T P + PA + C^T C - PB \left( D^T D \right)^{-1} B^T P = 0
\]

The two subproblem solutions are thus

1. The state feedback gain is \( K = - \left( D^T D \right)^{-1} B^T P \)
2. A switch metric is Quadratic Lyapunov function \( V = x^T P x \)
Consider the system
\[ \dot{x} = A_j x + B_j u \]
\[ z = C x + D u \]

Minimize the output energy
\[ \int_0^\infty z^T z \, dt \]

By solving the riccati equation
\[ A_j^T P_i + P_i A_j + C^T C - P_i B_j (D^T D)^{-1} B_j^T P_i < 0 \]

The two subproblem solutions are thus

1. The state feedback gain is \( K_i = - (D^T D)^{-1} B_i^* P_i \)

2. A switch metric is Common Quadratic Lyapunov function \( V_i = x^T P_i x \)
Linear Matrix Inequality (LMI)

- Iteratively divide $\mathcal{P}$ into subfamilies, one $C_i$ per subfamily
- 3 subfamilies imply 3 polytopic LMIs, $i \in \{1, 2, 3\}$
- Let $\theta_{ij}$ refer to subfamily $i$, vertex $j \in \{1, \ldots, v_i\}$
- Polytopic LMI is blkdiag $v_i$ LMIs in $\bar{A}_{ij} = \begin{bmatrix} A_{\theta_{ij}} & 0 \\ J_i C_{\theta_{ij}} & Z \end{bmatrix}$, $\bar{B}_{ij} = \begin{bmatrix} B_{\theta_{ij}} \\ 0 \end{bmatrix}$
- $\mathcal{P}$ has 3 subfamilies, $i = \{1..3\}$
- $C_1$ has 4 vertices, $j = \{1..4\}$

$$LMI_{C_1} = \begin{bmatrix} LMI_{\theta_{11}} & \cdots \\ \vdots & \ddots \\ \vdots & \cdots \\ LMI_{\theta_{14}} \end{bmatrix}$$
Switching

- Recall that $V = x^TPx$ is a scalar, e.g. $\sum$ Energy

- Switch logic uses this Lyapunov metric
  - By design at least one $C_i$ stabilizes $p(\theta^*) \in \mathcal{P}$
  - Each $C_i$ has a bound on the rate of decay of $V_i$

    If $V_i(\tau + dt) > \gamma V_i(\tau)$ for $dt > 0$, $0 < \gamma \leq 1$,

    Then reject $C_i$ & remove $i$ from set of feasible indices,

    Switch to next feasible index [8]

- Closed loop trial and error locates a stabilizing controller.
Find sufficiently many \( C_i \) to cover the unknown \( \theta^* \) by at least one \( C_i \).

\[ \begin{align*}
\dot{x} &= f(\theta^*, C_i) \\
v_i &= x^T P_i x \\
\frac{d}{dt} v_i &< 0 \text{ for some } C_i \\\n\text{If } v_i(\tau + dt) > \gamma v_i(\tau), \text{ Switch off } C_i
\end{align*} \]

Example Lyapunov function for \( \theta^* \) & \( C_1, P_1 \)
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Example: Aircraft Center of Gravity change

- Abstract the longitudinal aircraft dynamics of [4]:

\[
A_\theta = \begin{bmatrix}
\theta_1 & 0 & 1 \\
0 & 0 & 1 \\
\theta_2 & 0 & \theta_3
\end{bmatrix}, \quad B_\theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_\theta = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

where \( \{x_1, x_2, x_3\} \sim \{\alpha, \theta, q\} \). The relative degree of \( \alpha \) and the zero structure from \( \theta, q \) are preserved.

- Transfer Function:

\[
C_\theta (sI - A_\theta)^{-1} B_\theta = \frac{s^2 - s\theta_3 + \theta_2}{D(s, \theta_1, \theta_3)}
\]

- At \( \theta_2 = 0 \) the system is structurally unstable with
  - a transmission zero at \( s = 0 \),
  - a pole at the origin,
  - \( (A_\theta, B_\theta) \) uncontrollable.
Aircraft Example: Subfamilies

- In between the subfamilies, $\theta_2 \approx 0$, controllability and regulation are compromised.

- A single, structurally stable controller fails to regulate plants from both subfamilies.

- Solve two LMIs for the SFBK controller and Lyapunov matrix:
  - $\theta_2 \in [-4, -1] = \text{Top Branch: } K_T, P_T$
  - $\theta_2 \in [1, 4] = \text{Bottom Branch: } K_B, P_B$
Aircraft Example: Control Design

▶ Design weights $Q_Z = \text{diag}([1, 1, 25, 25]) / 10$ and $R_Z = 1$, Obtain:

$$K_B = [-3.5, -8.1, -2.6, -2.7] \quad P_B = \begin{bmatrix} 3.5 & 8.1 & 2.6 & 2.7 \\ 8.1 & 74. & 18. & 10. \\ 2.6 & 18. & 6.3 & 1.8 \\ 2.7 & 10. & 1.8 & 16. \end{bmatrix}$$


▶ Solve $\dot{x} = f(x_f, \theta, K, \vartheta) = 0$ with state, gain, and set point:

$$x = [x_1, x_2, x_3, \eta]^T, \quad K = [k_1, k_2, k_3, k_4], \quad \vartheta_r,$$

$$x_f = \begin{bmatrix} -\frac{\vartheta}{\theta_2} & \vartheta \left(1 + \frac{1}{\theta_2}\right) & 0 & \frac{\vartheta(\theta_1+k_1-k_2(1+\theta_2))}{k_4\theta_2} \end{bmatrix}^T$$
Aircraft Example: Simulation, Performance

Initial Conditions

\[
\begin{bmatrix}
  x_1(0) & x_2(0) & x_3(0) & \eta(0)
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
\]
\[
\vartheta(0) = -0.5
\]
\[
\sigma(0) = T
\]

\[Z = 0, \quad F = -1, \quad E^T = [0 \ 0 \ 1], \quad \theta^* = (\theta_1^*, \theta_2^*, \theta_3^*) = (-4, 1, -4)\]

\[
\frac{d}{dt} (x - x_f)^T P_i (x - x_f)
\]

Switch Logic
Aircraft Example: Simulation, Stability

Error Convergence

State Performance

Internal Stability
Future Work

1. Final State: Need to know $x_f$ to compute $\frac{\Delta V(t)}{\Delta t}$ where

$$V(t) = (x(t) - x_f)^T P (x(t) - x_f)$$

2. Switching Strategy: If the set of controllers is large, “trial and error” switch logic may perform poorly.

3. MMAR for a physically meaningful Aero Example
2. Switching Strategy: If $|C| \gg 1$, performance deteriorates

A Lyapunov switch logic accommodates System Identification to accurately select the next “in-the-loop” control.
Updated Plan of Study

1. Multiple Model Adaptive Regulation (MMAR)
2. Canonical Unfolding
3. Switching for MMAR
4. Extension to Nonlinear Systems

✓ MMAR for Different Zero Structures
  - Plant families are polytopes in the parameter space
  - Linear Matrix Inequalities (LMI) for Controller Design (1.)
  - Lyapunov Function based Switching (3.)

+ MMAR Improvements
  - Solve for Final State
  - Improve Switching Algorithm (3.)
  - MMAR for a more meaningful Aero example

× Canonical Unfolding and Invariant Forms (2.)
  - Enumerate plant subfamilies with Equivalence invariants
References


Questions or Comments?
Closed loop SISO system with plant, controller, and disturbance states,

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta} \\
\dot{\omega}
\end{bmatrix}
= \begin{bmatrix}
A_\theta + B_\theta K_1 & B_\theta K_2 & E \\
JC_\theta & Z & -JF \\
0 & 0 & Z
\end{bmatrix}
\begin{bmatrix}
x \\
\eta \\
\omega
\end{bmatrix}
\]

Consider the eigenvectors \((A_{cl} - \lambda I) \nu = 0\) for \(\lambda_z \in \text{spectrum } Z\)

\((Z, J)\) is controllable, so \(J\) has full column rank and right inverse \(JJ^\dagger = I\)

Row 2 of \(A_{cl}\) is \(J \begin{bmatrix}
C_\theta & J^\dagger (Z - \lambda_z) & -F
\end{bmatrix}\)

Eigenvectors \(\nu_z \subset \ker \begin{bmatrix}
C_\theta & 0 & -F
\end{bmatrix}\), the null space of \(e = C_\theta x - F\omega\)
Proposition on $B_* \in Co \{B_{ij}\}$

- **Proposition:** If $P$ exists to satisfy the $j$ inequalities:

$$A^T P + PA - PB_j R^{-1} B_j^T P - PB_j R^{-1} B_j^T P < -Q, \quad \forall j$$

Then any $B_* \in Co \{B_{ij}\}$ can be chosen for the state feedback gain

$$K = -R^{-1} B_*^T P$$

to satisfy the CQLF

$$A^T P + PA + K^T B_j^T P + PB_j K < -Q, \quad \forall j$$
Proposition on $B_* \in \text{Co}\{B_{ij}\}$

$\triangleright$ Proposition: If $P$ exists to satisfy the $j$ inequalities:

$$A^TP + PA - PB_jR^{-1}B_j^TP - PB_jR^{-1}B_j^TP < -Q, \quad \forall j \quad (1)$$

Then any $B_* \in \text{Co}\{B_{ij}\}$ can be chosen for the state feedback gain

$$K = -R^{-1}B_*^TP$$

to satisfy the CQLF

$$A^TP + PA + K^TB_j^TP + PB_jK < -Q, \quad \forall j$$

$\triangleright$ For the scalar case (1) is

$$2ap + q < 2p^2b_*b_j/r$$

which holds for $b_* \in [b_{\text{min}}, b_{\text{max}}]$. 
Proof of $B_* \in \text{Co} \{B_{ij}\}$

- Factor $(\Delta_j + I) B_j = B_*$ (Fix $\Omega_i$)

- Controllability: $\begin{bmatrix} I & A & A^2 & \ldots \end{bmatrix} (\Delta_j + I)^{-1} B_* \implies \Delta_j < I$

- Given: $A^T P + PA + K^T B_j^T P + PB_j K < -Q$, $\forall j$, $K = -R^{-1} B_j^T P$

- Then set $X = P^{-1}$, substitute $B_* = (\Delta_j + I) B_j$:

  $$B_* R^{-1} B_j^T + B_j R^{-1} B_*^T > X A^T + A X + X Q X$$

  $$(\Delta_j + I) B_j R^{-1} B_j^T + B_j R^{-1} B_j^T \left( I + \Delta_j^T \right) >$$

  $$B_j R^{-1} B_j^T + B_j R^{-1} B_j^T + \Delta_j B_j R^{-1} B_j^T + B_j R^{-1} B_j^T \Delta_j^T >$$

- The identity $A \succeq B$, $C \succeq D$, $A + C \succeq B + D$ implies

  $$B_j R^{-1} B_j^T + B_j R^{-1} B_j^T > \Delta_j B_j R^{-1} B_j^T + B_j R^{-1} B_j^T \Delta_j^T$$
Disturbance Estimation Regulator Design

Closed loop design equation is

\[
A_d = \begin{bmatrix}
A_\theta & B_\theta \bar{K}_i \\
-L_i C_\theta & \bar{A}_o + L_i \bar{C}_o + \bar{B}_o \bar{K}_i
\end{bmatrix}
\]

Yields a (non-convex) Bilinear Matrix Inequality

Were the observer an exact copy of the true plant dynamics, ie \( A_o = A_\theta \), the separation principle would hold and \( A_d \) could be transformed to an upper triangular form. Then solve for \( K_i \) in \( A_o + B_o K_i \) and again for the dual problem of \( L_i \) in \( \bar{A}_o^T + \bar{C}_o^T L_i^T \). In this idealized case a CQLF is certain for \( A_d \) in upper triangular form.
Normalized Uncertainty:

\[ T(P, C) = \left\| \begin{bmatrix} I \\ C \end{bmatrix} \cdot (I - PC)^{-1} \cdot \begin{bmatrix} P & I \end{bmatrix} \right\|_\infty \]

v-gap Metric:

\[ \delta_v(P, P_\theta) = \left\| (I + P_\theta P^*_\theta)^{-\frac{1}{2}} (P_\theta - P) (I + PP^*)^{-\frac{1}{2}} \right\|_\infty \]

Stability Assurance:

\[ T(P, C) \cdot \delta_v(P, P_\theta) < 1 \Rightarrow \text{Stability} \]

Moreover, for unstructured uncertainty

\[ T(P, C) \cdot \delta_v(P, P_\theta) \geq 1 \Rightarrow \text{Instability} \]
v-Gap Metric applied to “Aero” example

- Compensator includes Internal Model Dynamics. Plant includes SFBK:

- Fix $\theta_1, \theta_3$. Vary $\theta_2$. Recall $T(P, C) \cdot \delta_V(P, P_{z_1}) < 1 \Rightarrow$ Stability

- Wide variation of v-Gap metric conservatism vs. LMI for “aero” model
A Strict Equivalence transformation consists of 2 operations:

1. Similarity transformation induces the familiar Controllable and Observable decomposition

\[ \mathcal{H}_{co} \oplus \mathcal{H}_{\tilde{co}} \oplus \mathcal{H}_{\overline{co}} \oplus \mathcal{H}_{\overline{co}} \]

2. State feedback, Output Injection, and Scaling induce the decomposition:

\[ \mathcal{H}_{\infty} \oplus \mathcal{H}_{f} \oplus \mathcal{H}_{\eta} \oplus \mathcal{H}_{\varepsilon} \]

where

- ‘\( \infty \)’ are infinite divisors and describe the relative degree
- ‘\( f \)’ are finite divisors, ie invariant zeros
- \( \eta \) & \( \varepsilon \) are row and column indices and describe singular pencils
Strict Equivalence: State Space Interpretation

- Feedback and Scaling are implemented in the block diagram.

- Strict Equivalence is not related to stability.

- Relative Degree and Transmission Zeros are preserved.
A Strict Equivalence transformation consists of 2 operations

1. Similarity transformations of state:

\[
\begin{align*}
\bar{A} &= T^{-1}AT \\
\bar{B} &= T^{-1}B \\
\bar{C} &= CT \\
\bar{D} &= D
\end{align*}
\]

2. State feedback, output injection, and scaling \((D = 0)\):

\[
\begin{align*}
\hat{A} &= \bar{A} + L\bar{C} + BF \\
\hat{B} &= \bar{B}W \\
\hat{C} &= \bar{C} \\
\hat{D} &= 0
\end{align*}
\]
Strict Equivalence is categorized by Kronecker invariants:

<table>
<thead>
<tr>
<th>Regular Pencil</th>
<th>Singular Pencil</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite divisors, $f$</td>
<td>column indices, $\varepsilon$</td>
</tr>
<tr>
<td>infinite divisors, $\infty$</td>
<td>row indices, $\eta$</td>
</tr>
</tbody>
</table>

Regulator Design Structure differs from Strict Equivalence (S.E.):

1. S.E. Input/Output scaling abstracts needed structure
2. S.E. Singular pencil structure is not needed for control
Define a plant family with the form and parameter space:

\[ \mathcal{P} = \left\{ \frac{\theta_1}{s + \theta_2} \right\} \text{ such that } \theta_1, \theta_2 \in \mathbb{R} \]

Examine the structure at \( P_a, P_b, P_c \in \mathcal{P} \)

- \( \theta_1 \) can be less than, equal to, or greater than zero
- \( \theta_2 \) is greater than zero
Example: Strict Equivalence Structure & Regulation

Define a plant family with the form and parameter space:

\[ \mathcal{P} = \left\{ \frac{\theta_1}{s + \theta_2} \right\} \text{ such that } \theta_1, \theta_2 \in \mathbb{R} \]

Examine the structure at \( P_a, P_b, P_c \in \mathcal{P} \)

- \( \theta_1 \) can be less than, equal to, or greater than zero
- \( \theta_2 \) is greater than zero

Structure from Strict Equivalence:
\[
\left\{ \frac{1}{s}, \frac{0}{s} \right\}
\]

Structure of Regulation Sub-families:
\[
\left\{ \frac{1}{s}, \frac{-1}{s} \right\}
\]