Prime sum Graphs - Part 2

Bertrand Graphs

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Abstract

Bertrand graph is defined as a subgraph of a prime sum graph. Some properties of Bertrand graphs are studied with a list of problems.

PSGraph \( P_n \) restricted to the edges \( (x,y) \) such that \( x+y>n \) is called Bertrand Subgraph of \( P_n \) and is denoted by \( B_n \). Note that weights of the edges in \( B_n \) are primes \( p, n<p \leq 2n-1 \). PSGraph is a sparse graph and Bertrand graph being a subgraph of PSGraph is not only sparse but also disconnected for many values of \( n \).

Bertrand's postulate

Bertrand's postulate (now a theorem) states that for each \( n \geq 2 \) there is a prime \( p \) such that \( n < p < 2n \).

It was first proven by Pafnuty Chebyshev, and a short but advanced proof was given by Srinivasa Ramanujan. An elementary but involved proof by contradiction was due to Paul Erdős; the basic idea of the proof is to show that a certain binomial coefficient needs to have a prime factor within the desired interval in order to be large enough.

In terms of \( B_n \) an equivalent statement is:

**Bertrand's Postulate.** Bertrand subgraph is nonempty or there is at least one edge in \( B_n, n>1 \).

Fig. 1 \( B_n \) for \( n=4,6,8,10,12,14,16 \).

Degree Sequence

Define \( \sigma_b \)-function as, \( \sigma_b(i)=|\{x<i: x+i \text{ is a prime}\}| \). \( \sigma_b(i) \) represents the degree of the node \( i \) in \( B_n \). Note that, a pair \( (x,i) \) with \( x+i \) a prime is an edge in \( B_n \).

\((n,m)\)-PSGraph: \( m=\sum \sigma_b(i), i=1,...,n \).

Following Table shows the degree sequence for \( B_n, n \) even and \( n=4 \) to 32.
Connectivity of Bertrand Graph

Sparsity of edges in $B_n$, see also Fig. 1, shows that Bertrand graph $B_n$ may not always be connected. $B_n$ is connected assures a path between any two nodes. That is, for an arbitrary pair of nodes $u,v$ in $B_n$ there is a $u-v$ path with edge weights being primes.

**Lemma 1.** If $n+1$ is not prime then $B_n$ is disconnected.

$B_n$ is disconnected for many values. In fact, it contains isolated nodes. $B_6$ is disconnected consisting of 2 components: a 2-path and a 4-path.

**Lemma 2.** $B_n$ has a perfect matching if $n+1$ is prime.

That is, $B_n$ has a pairing of its nodes so that each pair sums to a prime.

**Theorem 3.** $B_n$ is connected if $n+1$ is prime and there exists a triplet $(x,x+1,x+2)$ such that $(x,x+1),(x+1,x+2)$ are edges in $B_n$ with the parallel edges spanning the nodes at least from $n$ to $n/2$.

As an example, $1,2,8,3,26,5,...,18,13,16,15,14,17,12,...,4,27,2$ is a Hamiltonian path in $B_{28}$, where $28+1=29$ is prime.

**Conjecture.** $B_n$ is connected iff $n\neq 6$ and $n+1$ is prime.

This is dependent on the following:

**Conjecture.** For $n>6$ there is at least one twin prime pair or a consecutive triple $(x,x+1,x+2)$ such that $2x+1$ and $2x+3$ are primes in $B_n$ between $n/2$ and $n$ where $n$ is such that $n+1$ is prime.

Table below shows $B_n$ for $n\leq 60$, $n$ even with $n+1$ prime and a triple as in Theorem 3.

<table>
<thead>
<tr>
<th>$n$</th>
<th>4</th>
<th>10</th>
<th>12</th>
<th>16</th>
<th>22</th>
<th>28</th>
<th>30</th>
<th>36</th>
<th>42</th>
<th>46</th>
<th>52</th>
<th>58</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$ Connected</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$B_n$ Connected</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>14</td>
<td>14</td>
<td>20</td>
<td>20</td>
<td>29</td>
<td>29</td>
<td>29</td>
<td>29</td>
<td>29</td>
<td>29</td>
</tr>
<tr>
<td>$B_n$ Connected</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>15</td>
<td>15</td>
<td>21</td>
<td>21</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$B_n$ Connected</td>
<td>1</td>
<td>7</td>
<td>10</td>
<td>10</td>
<td>16</td>
<td>16</td>
<td>22</td>
<td>22</td>
<td>31</td>
<td>31</td>
<td>31</td>
<td>31</td>
<td>31</td>
</tr>
</tbody>
</table>
Note that \( n=18 \) don't appear in the table. The pairs \((9,10)\) and \((11,12)\) represent cousin primes 19 and 23. The prime 19 is CP whereas 23 is friendly with 5 in \( B_{18} \). The edges of 23 together with the CP 19 form disjoint 1-2 and 3-4 paths viz., 1,18,5,14,9,10,13,6,17,2 and 3,16,7,12,11,8,15,4 spanning all the 18 nodes. The pair \((14,15)\) in addition makes the graph \( B_{18} \) connected.

**MATLAB**

The concept of **prime sum and prime power sum labelings** and respectively **prime sum and prime power sum graphs** were first conceived of by the authors during May 2011 while studying **coprime labelings and coprime graphs**. The prime sum concept bears direct relation with Bertrand’s Hypothesis.

The MATLAB code was written to generate these graphs using the geometry of \( n \)-th roots of unity. This aided in defining the circular representation of positive integers and the subsequent ideas of integer classification based on CRI.

Two representations were incorporated, viz.,

- The first one uses the base geometry of \( n \)-th roots of unity on a unit circle with the first point labelled 1 placed at \((1,0)\).
- The second one uses bipartition. For our purpose the bipartition with the labels partitioned into even \( \mathcal{E} \) and odd \( \mathcal{O} \) labels only are considered here.

A library of \( P_n \) and \( B_n \) for \( n=100 \) and above was generated and were useful to understanding and looking for symmetry or regularity.

**More Problems on \( P_n \)**

The even prime 2 does not appear as an edge weight in \( P_n \). Since the edge weights in \( P_n \) are primes it is relevant to ask questions like:

1. Does there exist cycles or paths with distinct edge weights? If so what is the longest cycle or path in \( P_n \). Investigate.
2. Does every cycle in \( P_n \) have a pair of parallel edges?
   The answer is no irrespective of parity of \( n \). Consider the cycle 1,4,7,6,1 or 2,3,4,7,6,5,2 in \( P_7 \) and the quadrangle 5,6,7,12,5 in \( P_{12} \). They are cycles with no parallel edges. Describe cycles with a parallel pair of edges.
3. Does there exist a cycle in \( P_n \) containing all odd primes \( \leq 2n \)?
   The answer is yes. For example, take \( n=10 \). The 10-cycle: 5,6,7,4,3,2,1,10,9,8,5 realises all primes \( \leq 20 \) as edge weights. Another example is, the 12-cycle 11,12,1,2,3,4,9,10,7,6,5,8,11 realises all primes \( \leq 24 \) as edge weights.
4. What happens if \( n \) is odd? It is known that \( P_n \) for \( n \) even is Hamiltonian and \( P_{n+1} \) has a Hamiltonian path.
References


Abstract
In 1845 Bertrand postulated that there is always a prime between \( n \) and \( 2n \), and he verified this for \( n < 3,000,000 \). Tchebychev gave an analytic proof of the postulate in 1850. In 1932, in his first paper, Erdős gave a beautiful elementary proof using nothing more than a few easily verified facts about the middle binomial coefficient. We also describe a result of Greenfield and Greenfield that links Bertrand’s postulate to the statement that \( \{1, \ldots, 2n\} \) can always be decomposed into \( n \) pairs such that the sum of each pair is a prime.

**Theorem 1.2** For \( n > 0 \), the set \( \{1, \ldots, 2n\} \) can be partitioned into pairs \( \{a_1, b_1\}, \ldots, \{a_n, b_n\} \) such that for each \( 1 \leq i \leq n \), \( a_i + b_i \) is a prime.

2. **On integral sum graphs**

Harary introduced **sum graphs on integers** which are now known as **Integral Sum Graphs**. Since then several papers have appeared on the concept.


Abstract
We introduced the sum graph of a set \( S \) of positive integers as the graph \( G(S) \) having \( S \) as its node set, with two nodes adjacent whenever their sum is in \( S \). Now we study sum graphs over all the integers so that \( S \) may contain positive or negative integers on zero. A graph so obtained is called an integral sum graph. The sum number of a given graph \( G \) was defined as the smallest number of isolated nodes which when added to \( G \) result in a sum graph. The integral sum number of \( G \) is analogous. We see that all paths and all matchings are integral sum graphs. We find the integral sum number of the small graphs and offer several intriguing unsolved problems.

3. **Open Problems - Graph Theory and Combinatorics** collected and maintained by Douglas B. West, [http://www.math.uiuc.edu/~west/](http://www.math.uiuc.edu/~west/)

**Combinatorial Gray codes**
Named for the classical Gray code listing binary vectors (cyclically) with one bit change between successive vectors, A combinatorial Gray code is a listing of the objects in a set using only specified changes between successive objects. The last item should also be close to the first, so what is sought is a Hamiltonian cycle in the graph defined by the permitted adjacencies.

- **Revolving Door (Middle Levels) Conjecture** (there is a cycle through the subsets of \( \mathbb{Z}_{2k+1} \) with sizes \( k \) and \( k+1 \) by adding or deleting one element at each step)
- **Traversal by Prime Sum** (for \( m \geq 2 \), does the graph with vertex set \( \mathbb{Z}_{2m} \) and edges joining numbers whose sum is prime always have a Hamiltonian cycle?)

**Question:** Let \( G_m \) be the graph with vertex set \( \{1, 2, 3, \ldots, 2^m\} \) such that \( x \) is an edge if and only if \( x+y \) is prime. Is \( G_m \) Hamiltonian when \( m \geq 2 \)?

**Comments/Partial results:** It is easy to build a Hamiltonian cycle when \( 2m+1 \) and \( 2m+3 \) are both prime, but it is not even known if \( G_m \) is Hamiltonian for infinitely many \( m \).

**References:** This question was discussed in a thread on the now-defunct mailing list COMB-L. [http://sci.tech-archive.net/Archive/sci.math/2008-05/msg00047.html](http://sci.tech-archive.net/Archive/sci.math/2008-05/msg00047.html)

**Circular Prime Sums**, *From: Albert, Date: Wed, 30 Apr 2008 23:47:10 -0700 (PDT)*, On May 1, 10:55 am, Gerry Myerson.