

Lecture 3

Bond Yields and Term Structure of Interest Rates

1. Basic Concepts

A. *Zero-coupon Bonds (Discount Bonds)*- It makes a single payment at the maturity date. The size of the payment is the face value of the bond. For an n -period pure discount bond price $P_{n,t}$ and n -period bond rate $r_{n,t}$, and single payment of face value (maturity value) F , we have:

$$\begin{aligned}
 P_{1,t} &= F/(1+r_{1,t}), \\
 P_{2,t} &= F/(1+r_{2,t})^2, \\
 &\cdot \\
 &\cdot \\
 P_{n,t} &= F/(1+r_{n,t})^n. \text{ To simplify, we may set } F=\$1, \text{ so} \\
 P_{n,t} &= 1/(1+r_{n,t})^n.
 \end{aligned} \tag{3.1.1}$$

The rate of return can be obtained from price as:

$$(1+r_{n,t}) = P_{n,t}^{-\left(\frac{1}{n}\right)} \tag{3.1.2}$$

Expressing in natural logarithm:

$$\ln P_{n,t} = -n \ln(1+r_{n,t}), \quad \text{with } \ln F = \ln 1 = 0 \text{ by setting } F = \$1.$$

$$P_{n,t} = e^{-n \ln(1+r_{n,t})} \tag{3.1.4}$$

$$r_{n,t} = -\left(\frac{1}{n}\right) p_{n,t} \tag{3.1.5}$$

where $r_{n,t} \cong \ln(1+r_{n,t})$ and $p_{n,t} = \ln P_{n,t}$. log yield and log price (expressed in lower case) are commonly used in the empirical analysis.

B. Coupon Bonds

For an n -period coupon paying bond with the spot rates $r_{i,t}$, we have the bond price as:

$$P_{n,t} = \frac{C_{t+1}}{(1+r_{1,t})} + \frac{C_{t+2}}{(1+r_{2,t})^2} + \dots + \frac{C_{t+n}}{(1+r_{n,t})^n} + \frac{F_{t+n}}{(1+r_{n,t})^n} = \sum_{i=1}^n V_i \quad (3.1.6)$$

$$V_i = \frac{C_{t+i}}{(1+r_{i,t})^i}, i = 1, 2, \dots, (n-1) \text{ and } V_n = \frac{C_{t+n} + F_{t+n}}{(1+r_{n,t})^n}.$$

To simplify, we set $C_{t+i} = C$ and $F_{t+n} = I$. Thus,

$$P_{n,t} = \frac{C}{(1+r_{1,t})} + \frac{C}{(1+r_{2,t})^2} + \dots + \frac{C+1}{(1+r_{n,t})^n} \quad (3.1.7)$$

- ◆ If $P_{n,t} = I$, the bond is selling at par ($F_{t+n} = I$ as well), the *coupon rate* ($r_c = C/F_{t+n}$) is equal to the market rate.
- ◆ If $n \rightarrow \infty$ (the bond is perpetuity), then $r = \frac{C}{P}$.
- ◆ Bond prices move inversely to changes of rate of interest or yield to maturity.
- ◆ Prices of long-term bonds are more sensitive to a given change in interest rate than the prices of short-term bond.

C. Holding Period Yield (HPY)

The total return for holding one-period coupon paying bond between t and $t+1$ is the sum of the capital gain and coupon payment. Thus, one-period holding period yield, $H_{n,t+1}$, is given by:

$$H_{n,t+1} = \frac{P_{n-1,t+1} - P_{n,t} + C_t}{P_{n,t}} \quad (3.1.8)$$

In this formula, the n -period bond becomes an $(n-1)$ period bond after one period. Thus, P_n is an n -period bond price at time t , and $P_{n-1,t+1}$ is the price of an $(n-1)$ -period bond when it is sold at time $t+1$.

2. Bond Price Volatility - Duration and Convexity

Since in most of fixed income investments the interests earned are compound over time, the yield to maturity (*YTM*), y , is usually used to discount the coupon payments (C) and the face value ($F=1$). Due to the fact that coupon payments are made before maturity, the coupon components are equivalent to a discount bond. Thus, the Macaulay duration concept is more appropriate to describe the valuation of bond. Also, the duration provides an approximate change in bond price for a small change in yield. Let us rewrite the bond price formula as follows:

$$P = \frac{C}{(1+r_{1,t})} + \frac{C}{(1+r_{2,t})^2} + \dots + \frac{1+C}{(1+r_{n,t})^n} \text{ (We drop the subscript of } P \text{ to simplify)}$$

$$= \frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \dots + \frac{1+C}{(1+y)^n}, \quad y = r_{1,t} = r_{2,t} = \dots = r_{n,t}. \quad (3.2.1)$$

A. Duration

Taking the first derivative of P w.r.t. y , we obtain:

$$\frac{dP}{dy} = \frac{(-1)C}{(1+y)^2} + \frac{(-2)C}{(1+y)^3} + \dots + \frac{(-n)(1+C)}{(1+y)^{n+1}}, \quad (3.2.2)$$

$$\frac{dP}{dy} \cdot \frac{1}{P} = -\frac{1}{(1+y)} \cdot \left[\frac{1C}{(1+y)} + \frac{2C}{(1+y)^2} + \dots + \frac{n(1+C)}{(1+y)^n} \right] \cdot \frac{1}{P} \quad (3.2.3)$$

$$D = \text{Macaulay Duration} = \left[\frac{1C}{(1+y)} + \frac{2C}{(1+y)^2} + \dots + \frac{n(1+C)}{(1+y)^n} \right] \cdot \frac{1}{P} \quad (3.2.4)$$

$$= \left[C \sum_{i=1}^n \frac{i}{(1+y)^i} + \frac{n}{(1+y)^n} \right] \cdot \frac{1}{P} \quad (3.2.5)$$

- ◆ *Macaulay Duration* is [a weighted average of the maturities of the underlying discount bonds], where the weight on each maturity is the present value of the corresponding discount bond calculated using the coupon bond's yield as the

discount rate. (Numerical examples can be found in *Bond Markets, Analysis and Strategies*, F. Fabozzi and T.D. Fabozzi, 1989).

- ◆ When $C = 0$, the bond is a discount bond and Macaulay's duration equals maturity. When $C > 0$, Macaulay's duration is less than maturity and it declines with the coupon rate.

- ◆
$$\text{Modified Duration} = - \frac{1}{(1+y)} \cdot \frac{C \sum_{i=1}^n \frac{i}{(1+y)^i} + \frac{n}{(1+y)^n}}{P} = - \frac{1}{(1+y)} \cdot D \quad (3.2.6)$$

- ◆
$$\frac{dP}{dy} \cdot \frac{1}{P} = - \text{Modified Duration} \quad (3.2.7)$$

It says that modified duration is related to the approximate percentage change in bond price for a given change in yield. If the modified duration is 10, an increase in the yield of 1 basis point (say from 4.00% to 4.01%) will cause a 10 basis point or 0.10% decline in the bond price.

- ◆ Note that if the cash flows occur m times per year, the durations are adjusted by dividing by m as: Duration in years = Duration in m periods per year / (m).

B. Convexity

Using the first two terms of a Taylor expansion series, the price change can be approximated as:

$$dP = \frac{dP}{dy} dy + \frac{1}{2} \frac{d^2P}{dy^2} (dy)^2 + \text{Error} \quad (3.2.8)$$

$$\frac{dP}{P} = \left[\frac{dP}{dy} \frac{1}{P} \right] \cdot dy + \left[\frac{1}{2} \frac{d^2P}{dy^2} \cdot \frac{1}{P} \right] \cdot (dy)^2 + \text{Error} \quad (3.2.9)$$

$$\frac{dP}{P} \cong [- \text{modified duration}] dy + [\text{Convexity}] (dy)^2 \quad (3.2.10)$$

$$\text{Convexity} = \frac{1}{2} \frac{d^2 P}{dy^2} \cdot \frac{1}{P} \quad (3.2.11)$$

The percentage price change due to *convexity* is:

$$\frac{dP}{P} = (\text{Convexity})(dy)^2 \quad (3.2.12)$$

To compute the convexity, the second derivative of the price equation is:

$$\frac{d^2 P}{dy^2} = \sum_{i=1}^n \frac{i(i+1)C}{(1+y)^{i+2}} + \frac{n(n+1)}{(1+y)^{n+2}} \quad (3.2.13)$$

Note that if the cash flows occur m times per year, convexity is adjusted to an annual figure as: Convexity in years = Convexity in m periods per year / (m^2).

3. Term Structure of Interest Rates

A. Definition

- ◆ *Term structure of interest rates* : Description of the yields to maturity on zero-coupon bonds that have different maturities.
- ◆ *Yield curve*: The schedule that shows the relationship between the maturity of bonds in the same risk class and the yields on those bonds.

B. Some Empirical Observations on Yields

- ◆ Short-term yields have been considerably more volatile than long-term yields.
- ◆ Ascending yield curves are formed when both short-and long-yields are relatively low.
- ◆ On average, long-term yield is higher than the short- term yield.

C. Interpretations of Yield Curve

Let $E(H)$ = expected holding period yield

$$= (C + P^e - P)/P \quad (3.3.1)$$

$$= r + r/r^e - 1 \quad (3.3.2)$$

where $r/r^e - 1 = g$ (capital gains if $g > 0$),

R = long bond yield,

r = short bond yield.

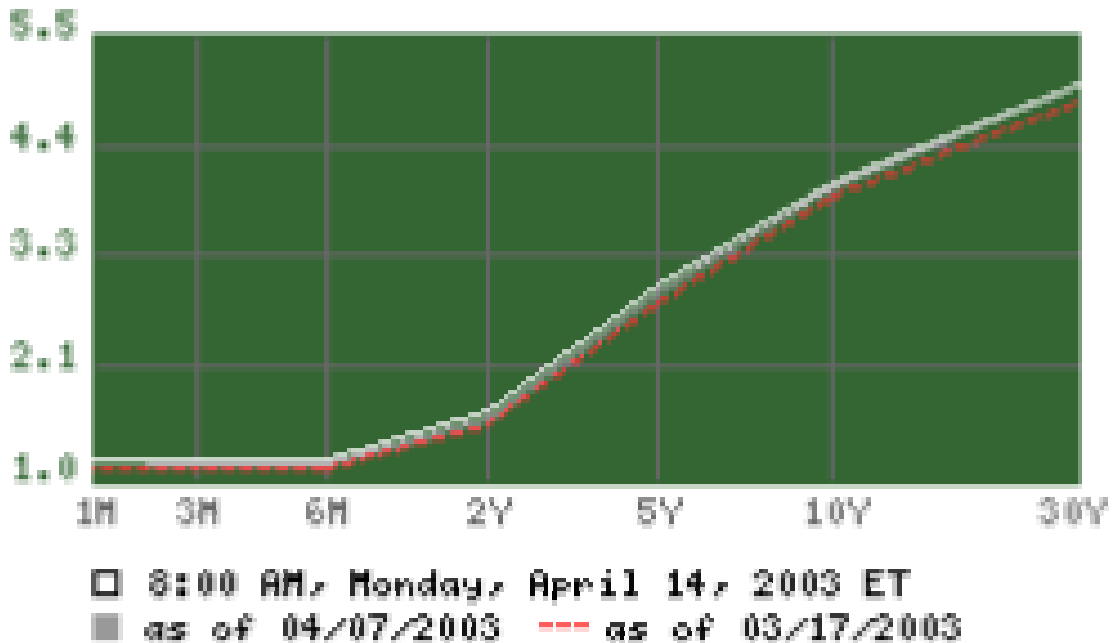
◆ *Interpretation of the yield curve*

When r^e rises in the future, then $(r/r^e - 1) < 0$ and $g < 0$.

Demand for long-term bond declines \Rightarrow Sell long term bonds and buy short term bonds $\Rightarrow P \downarrow$ for LT bonds and $P \uparrow$ for ST bonds $\Rightarrow R \uparrow, r \downarrow$.

Go to the website of the Bond Market Association: <http://www.investinginbonds.com/> and read in the section entitled, "The Investor's Guide to Bond Basics." You find the yield curve displayed below.

Current Treasury Yield Curve



How do you interpret this yield curve? What is the economic and financial implication of the yield curve?

4. Interest Rate Spread and Risk –Theoretical Exposition

Assuming there is a single representative agent with preferences

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (3.4.1)$$

$$\text{Subject to the condition: } c_t + B_{1,t} + B_{2,t} \leq d_t + B_{1,t-1} r_{1,t-1} + B_{2,t-2} r_{2,t-2} \quad (3.4.2)$$

where $B_{j,t}$ is the amount lent for j periods by buying assets at time t , $r_{j,t}$ the gross rate of returns for holding j ($j = 1,2$) period loans (bonds). The $r_{j,t}$ is known and risk free at the beginning of t . Thus, at time t , $r_{1,t}^{-1}$ and $r_{2,t}^{-1}$ are the prices of a perfectly sure claim to one unit of consumption at time $t+1$ and $t+2$, respectively. (Note we switch the notation in this section by using $r_{2,t}$ as the long-term (two-period) bond return and $r_{1,t}$ as short-term (one-period) bond return). The dividend d_t evolves according to a Markov Process, which is available of the consumption good per capita for each t , and $c_t \leq d_t$. By setting up Lagrangian:

$$J = E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) + \lambda_t (d_t + B_{1,t-1} r_{1,t-1} + B_{2,t-2} r_{2,t-2} - c_t - B_{1,t} - B_{2,t})]. \quad (3.4.3)$$

It makes clear by writing out the above equation as:

$$\begin{aligned} J = & E_0 \{ [\beta^0 u(c_0) + \beta^1 u(c_1) + \dots + \beta^t u(c_t) + \beta^{t+1} u(c_{t+1}) + \dots + \beta^\infty u(c_\infty)] + \\ & \beta^0 \lambda_0 (d_0 + B_{1,0-1} r_{1,0-1} + B_{2,0-2} r_{2,0-2} - c_0 - B_{1,0} - B_{2,0}) + \dots + \\ & \beta^t \lambda_t (d_t + B_{1,t-1} r_{1,t-1} + B_{2,t-2} r_{2,t-2} - c_t - B_{1,t} - B_{2,t}) + \\ & \beta^{t+1} \lambda_{t+1} (d_{t+1} + B_{1,t} r_{1,t} + B_{2,t-1} r_{2,t-1} - c_{t+1} - B_{1,t+1} - B_{2,t+1}) + \\ & \beta^{t+2} \lambda_{t+2} (d_{t+2} + B_{1,t+1} r_{1,t+1} + B_{2,t} r_{2,t} - c_{t+2} - B_{1,t+2} - B_{2,t+2}) + \dots \} \quad (3.4.3)' \end{aligned}$$

The first order conditions are:

$$\begin{aligned}\frac{\partial J}{\partial c_t} &= \beta^t \frac{\partial u(c_t)}{\partial c_t} - \beta^t \lambda_t \frac{\partial c_t}{\partial c_t} = 0 \\ \frac{\partial J}{\partial B_{1,t}} &= -\beta^t \lambda_t \frac{\partial B_{1,t}}{\partial B_{1,t}} + E_t \beta^{t+1} \lambda_{t+1} r_{1,t} \frac{\partial B_{1,t}}{\partial B_{1,t}} = 0 \\ \frac{\partial J}{\partial B_{2,t}} &= -\beta^t \lambda_t \frac{\partial B_{2,t}}{\partial B_{2,t}} + E_t \beta^{t+2} \lambda_{t+2} r_{2,t} \frac{\partial B_{2,t}}{\partial B_{2,t}} = 0\end{aligned}$$

With $\beta^t \neq 0$, the first order conditions become:

$$u'(c_t) - \lambda_t = 0, \quad \lambda_t = u'(c_t), \lambda_{t+1} = u'(c_{t+1}), \lambda_{t+2} = u'(c_{t+2}), \quad (3.4.4)$$

$$-\lambda_t + \beta E_t \lambda_{t+1} r_{1,t} = 0, \quad (3.4.5)$$

$$-\lambda_t + \beta^2 E_t \lambda_{t+2} r_{2,t} = 0 \quad (3.4.6)$$

where $\{\lambda_t\}$ is a sequence of random Lagrangian multipliers. Substituting $\{\lambda_{t+j}\}$,

$$E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot r_{1,t} \right] = 1 \quad (3.4.7)$$

$$E_t \left[\beta^2 \frac{u'(c_{t+2})}{u'(c_t)} \cdot r_{2,t} \right] = 1 \quad (3.4.8)$$

Since $r_{1,t}$ and $r_{2,t}$ are known with certainty at the beginning of t , these two equations can be written as:

$$r_{1,t}^{-1} = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] \quad (3.4.9)$$

$$\begin{aligned}r_{2,t}^{-1} &= E_t \left[\beta^2 \frac{u'(c_{t+2})}{u'(c_t)} \right] \\ &= E_t E_{t+1} \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} \right] \quad (\text{The law of iterated expectations})\end{aligned}$$

$$= E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] \cdot E_{t+1} \left[\beta \frac{u'(c_{t+2})}{u'(c_{t+1})} \right] = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] \cdot r_{1,t+1}^{-1} \quad (3.4.10)$$

Using $E(XY) = E(X)E(Y) + Cov(XY)$, (3.4.11)

$$r_{2,t}^{-1} = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] \cdot E_t r_{1,t+1}^{-1} + Cov \left[\beta \frac{u'(c_{t+1})}{u'(c_t)}, r_{1,t+1}^{-1} \right] \quad (3.4.12)$$

$$r_{2,t}^{-1} = r_{1,t}^{-1} \cdot E_t r_{1,t+1}^{-1} + Cov \left[\beta \frac{u'(c_{t+1})}{u'(c_t)}, r_{1,t+1}^{-1} \right] \quad (3.4.13)$$

This is a generalized version of the expectations theory of the term structure of interest rates. Several points merit comment:

- (1) The risk premium is contained in the covariance term – the covariance of the discounted marginal rate of substitution of consumption and the future asset return. If $Cov(\cdot) = 0$, then the pure expectations hypothesis holds. (Note we use $r_{2,t}$ to denote the long-term bond return in this section).

$$r_{2,t}^{-1} = r_{1,t}^{-1} \cdot E_t r_{1,t+1}^{-1} \quad (3.4.14)$$

- (2) If the utility is linear in consumption such that $\frac{u'(c_{t+1})}{u'(c_t)} = 1$,

$$E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot r_{1,t} \right] = 1 \quad \text{leads to:}$$

$$\beta = \frac{1}{r_{1,t}}, \quad \text{where } r_{1,t} \text{ is constant.}$$

- (3) More generalized model can be achieved by adding stocks as an asset into the model. Moreover, you may incorporate the foreign bonds as well as foreign stocks into the model (see Chiang and Trinidad, *IEJ*, 1997).

The above model is known as Locus model (see “The Term Structure of Interest Rates,” in *Dynamic Macroeconomic Theory*, T. Sargent, 1987, 102-108).

5. Alternative Theories of Term Structure of Interest Rates

A. The Unbiased Expectation Theory (UET)

This theory explains the term structure of interest rates by interest rate expectations. In particular, longer-term rates will be above the current short rate if rates are expected to rise in the future and will be below if rates are expected to decline.

Unbiased Forward Rate Hypothesis

$$(1 + R_{n,t}) = [(1 + R_{1,t})(1 + r_{1,t+1}^e) \dots (1 + r_{1,t+n-1}^e)]^{1/n}, \quad (3.5.1)$$

$$(1 + R_{n,t}) = [(1 + R_{1,t})(1 + f_{1,t+1}) \dots (1 + f_{1,t+n-1})]^{1/n}, \quad (3.5.2)$$

$$\text{where } r_{1,t+j}^e = f_{1,t+j}, j = 1, \dots, n-1, \quad (3.5.3)$$

$$f_{1,t+n-1} = (1 + R_{n,t})^n / (1 + R_{n-1,t})^{n-1} - 1 \quad (3.5.4)$$

The best predictor of the future interest rates can be calculated from today's yield curve.

Notations:

1st subscript = the maturity of bond; 2nd subscript = the time period for which the rate is applicable; $f_{1,t+1}$ = *implicit forward rate*;

$r_{n,t+j}^e$ = expected n period return for j year;

$R_{n,t}$ = the geometric average of the n -period of expected interest rates at time t .

B. Liquidity Premium Theory

Short term bonds are more desirable than long term bonds for two reasons: (i) high liquidity; (ii) less risk resulted from price uncertainty.

Relaxing the assumption in (3.5.3) by imposing *uncertainty equivalent*, we write:

$$f_{1,t+j} = r_{1,t+j}^e + \phi_{1,t+j}, \quad (3.5.5)$$

$\phi_{1,t+j}$ = risk premium at time $t+j$. Substituting (3.5.5) into (3.5.2):

$$(1 + R_{n,t}) = [(1 + R_{1,t})(1 + r_{1,t+1}^e + \phi_{1,t+1}) \dots (1 + r_{1,t+n-1}^e + \phi_{1,t+n-1})]^{1/n} \quad (3.5.6)$$

where $0 < \phi_{1,t+1} < \dots < \phi_{1,t+n-1}$.

C. Market Segmentation Theory (Hedging pressure theory)

Imperfect substitutability between long and short bonds. Short and long rates are segmented and interest rates depend on respective supply and demand conditions.

6. Continuous time Stochastic Models (Santomero and Babbel, 2001, pp.134-137)

A. One Factor Model in an uncertainty environment

Continuous time one factor model: It is assumed that information structure is Markov in the currently prevailing short rate, which is assumed to capture all currently available information relevant for pricing default-free bonds, $P(r,t,T)$ denotes the price at t of a zero coupon bond maturing at T with a face value of \$1, given that the currently prevailing short rate is r . The evolution of interest rate is assumed to follow a diffusion process:

$$dr = \mu(r,t) dt + \sigma(r,t)dZ, \quad (3.6.1)$$

Here $\mu(r,t)$ measures the expected change in the interest rate per unit time, $\sigma(\cdot)$ measures the standard deviation of changes in the interest rate per unit time, dz is the increment to a Wiener process.¹

A specific form can be written as:

$$dr = k(\mu - r)dt + \sigma r^\gamma dz, \quad (3.6.2)$$

where k = the mean reversion adjustment coefficient, μ = mean value of short-term interest rate. This model assumes that there is only one source of risk and is referred to as a one-factor model. This implies that all bond prices depend on the movements of the rate (r). Empirically, the dominant factor, the level of movement, can explain much of the yield curve movements. This lead to the CIR model below.

Cox, Ingersoll and Ross (1985) Model: if $\gamma = 1/2$

$$dr = k(\mu - r)dt + \sigma r^{\frac{1}{2}} dz, (k > 0). \quad (3.6.3)$$

The interest rate in (3.6.3) is attracted elastically toward its mean value μ and is influenced by a noise term whose variance is proportional to the prevailing level of interest rate. A higher interest rate appears to be more volatile. CIR defines market price of risk, λ , as the excess return required for the risk factor per unit risk measure. In expression,

$$\lambda(r,t) = \frac{\mu - r}{\sigma},$$

where $\lambda(r,t)$ depends on the risk factor, r , and time, t . The excess return $(\mu - r)$ is the premium for bond returns, and is called the term premium. Vasicek replaces $(\mu - r)$ by $(R-r)$, which is given by:

Vasicek Model (1977) if $\gamma=0$, $\mu = R$,

$$dr = k(R - r)dt + \sigma dz \quad (3.6.4)$$

Equation (3.6.4) assumes that all bond prices are driven by the short-term interest rate. This short-term interest rate seeks to converge to a fixed long-term rate, R . If the rate goes too high, it will fall back down to the constant long rate. If $\sigma = 0$, then (3.6.4) becomes standard model.

Lattice Models: If changes in spot interest rate are normally distributed, then the evolution of the spot interest rate can be described by :

$$\Delta r(t) = [a(t) - b(t)r(t)]h + \sigma(t)\Delta z \quad (3.6.5)$$

where $\Delta r(t) \equiv r(t+h) - r(t)$, h = the length of the time interval, $\sigma(t)$ = the volatility at date t , Δz is a normally distributed random variable with zero mean and variance h , The parameters $a(t)$, $b(t)$, and $\sigma(t)$ are deterministic functions of time and independent of the spot rate. (Jarrow and Turnbull, 1996, pp.463-472).

The process follows the binomial representation as:

$$r(t+h) - r(t) = \begin{cases} [a(t) - b(t)r(t)]h + \sigma(t)\sqrt{h}; & \text{probability } 1/2 \\ [a(t) - b(t)r(t)]h - \sigma(t)\sqrt{h}; & \text{probability } 1/2. \end{cases} \quad (3.6.6)$$

B. *Two-Factor Model (Brennan and Schwartz, 1982):* Interest rate models assert that the shape of the yield curve depends on the short rate and long rate (Two-factor model). Note that In predicting short rate changes, the second factor can be inflation rate or volatility.

$$dr = k(R - r)dt + r\sigma_1 dz \quad (3.6.7)$$

$$dR = (hr + gR)dt + R\sigma_2 dw \quad (3.6.8)$$

Where σ_1 and σ_2 are the standard deviations of the short rate, r , and long rate, R respectively; k , h , g are estimated from the historical data.

The empirical estimation is given by Longstaff and Schwartz (1992, *J.Finance*, 47, 1259-1282).

Applications and example can be found in T.S.Y.Ho and S.B. Lee, *The Oxford Guide to Financial Modeling*, Oxford University Press, 2004, pp.123-171).

Concluding Remarks:

- ◆ The continuous-time no-arbitrage method provides a fully specified model of bond prices for empirical work.
- ◆ It gives an explicit measure of the risk characteristics of the priced assets.

References:

Robert Jarrow and Stuart Turnbull, *Derivative Securities*, South-Western Pub, ITP, 1996.

Thomas S.Y. Ho and Sang Bin Lee, *The Oxford Guide to Financial Modeling*, Oxford University Press, 2004.

Endnotes

1. The price of zero bonds with some mathematical manipulation becomes (see J.Ingersoll in *Finance* ed. by Eatwell et al., 1989, p.172-179):

$$P(r,t,T) = E \left\{ \exp \left[- \int_t^T r(s) ds \right] \right\}$$

Appendix 1 Illustration for Duration

A. The Formula of Duration

Duration (D) is the weighted average time to maturity on a financial security using the relative present values of the cash flows (*C*) as weights.

$$D = \frac{\sum_{i=1}^n PV_i \cdot (i)}{\sum_{i=1}^n PV_i} = \frac{\sum_{i=1/m}^n \frac{C \cdot (i)}{(1 + r/m)^{mi}}}{\sum_{i=1/m}^n \frac{C}{(1 + r/m)^{mi}}} \quad (\text{A-1})$$

where *D* = duration measured in years

C = cash flow received on the security at end of period *t*

PV_i = present value of the cash flow received at the end of the period *t*

r (= *y*) annual yield to maturity (The text uses *y*)

i = 1 to *n*, the period in which a cash flow is received

m = Number of times per year interest is paid.

To understand the concept implied by the definition, think of an *n* year annual payment coupon bond as a portfolio of *n* zero coupon bonds where the first *n-1* of the bonds pay the coupon amount and the last one pays the coupon plus the par amount. The coupon bond's duration is the average maturity of the zero coupon bonds. However, we cannot take a simple average because not all the cash flows in each year are identical. Moreover, we cannot compare a cash flow in year 1 with a cash flow in year *n*. So we construct a series of weights that tell us what percentage of our money (in today's dollars) we recover in each year.

To do this, take the present value of each cash flow divided by the purchase price of the bond. For instance, for a 5 year bond suppose we recover in present value terms 5% of our investment in year one, 4% in year two, 3% in year three, 2% in year four and 86% in year five. We receive cash in each of the next five years (or a zero matures in each of the next five years): 1, 2, 3, 4 and 5. Thus, the bond's duration is the weighted average maturity on the portfolio of zero-coupon bonds. That is,

$$D = \frac{\sum_{i=1}^5 PV_i \cdot (i)}{P} = 5\% \times (1) + 4\% \times (2) + 3\% \times (3) + 2\% \times (4) + 86\% \times (5) = 4.6 \text{ years.}$$

On the time value of money basis, the initial investment in the 5-year bond is recovered after 4.6 years. After that time, the investor earns a profit on the bond. You may think of Duration as a weighted average of the times in which cash is received, a slightly different wording than the more common definition above.

Example: The Duration of a Four Year Bond

A four year bond with a coupon rate of 10% paid semiannually contains the following information:

$r = 8\%$,

$r_c = \text{coupon rate} = 10\%$,

$C = \text{Periodic cash flow in dollars} = (\$1,000 \times 10\%)/2 = \$50$

$n = \text{Number of compounding or payment periods} = 8$ (The increment of $i = 1/2$)

$F_{t+n} = \text{Maturity Value or Terminal} = \$1,000.00$

$m = 2$

$D = \text{Duration}$

$P = FPV = \text{Fair Present Value} = \text{Current Price} = \$1,067.34$

$$D = \frac{\sum_{i=1/m}^N \frac{C \times i}{(1+r/m)^{mi}}}{P} = \frac{\sum_{i=1/2}^{3\frac{1}{2}} \left[\frac{\$50 \times i}{(1+8\%/2)^{2i}} \right] + \frac{(\$1,000 + \$50) \times 4}{(1+8\%/2)^{2i}}}{\$1067.34} = 3.42 \text{ years}$$

where $t = \{1/2, 1, 1\frac{1}{2}, 2, \dots, 3\frac{1}{2}\}$. Note that the increment of index $i = \frac{1}{2}$, rather than 1 as standard case.. Writing this expression in a long form, we obtain:

$$D = \frac{\$50 \times 0.5}{(1+4\%)^1} + \frac{\$50 \times 1}{(1+4\%)^2} + \frac{\$50 \times 1.5}{(1+4\%)^3} + \frac{\$50 \times 2}{(1+4\%)^4} + \frac{\$50 \times 2.5}{(1+4\%)^5} + \frac{\$50 \times 3}{(1+4\%)^6} + \frac{\$50 \times 3.5}{(1+4\%)^7} + \frac{\$1,050 \times 4}{(1+4\%)^8}$$

$$= 0.02 + 0.04 + 0.06 + 0.08 + 0.10 + 0.11 + 0.13 + 2.88 = 3.42.$$

It is can be seen that the initial investment is recovered as each coupon payment is received. Particular, the percent of initial investments is recovered from 0.02, 0.04, 0.06,..., 2.88. [The duration on this 4-year bond is 3.42 years.](#)

B. Features of Duration

- (a) *Duration and coupon interest.* Duration is a time measure, usually presented in years. The greater the coupon the shorter the duration. With a greater coupon the percentage weights on the early years are increased, thus reducing the average maturity.
- (b) *Duration and maturity.* The duration of a zero coupon bond is its maturity since it has a 100% weight on the terminal cash flow. The longer the maturity the longer the duration. It can be shown that duration increase with the maturity of a bond.
- (c) *Duration and yield to maturity.* Although duration is a modified measure of maturity, it is still a measure of maturity and maturity is the dominant determinant. The higher the promised yield to maturity the shorter the duration. Using a higher interest rate decreases the percentages weights on more distant cash flows (because of the compounding effect the present value of more distant cash flows drops (%) more than the present value of near term cash flows).

C. Economic Meaning of Duration

Taking the partial derivative of the bond price formula with respect to interest rates for a zero coupon bond yields a simple relationship:

$$\frac{\Delta P / P}{\Delta r / (1 + r)} = -D \quad (\text{Here we use the discrete form in that } \Delta P = dP, \Delta r = dr) \quad (\text{A-2})$$

Duration can be used to predict the % price change for small interest rate changes when there are no coupons (the price elasticity of interest rate change). This equation does not work for coupon bonds; its use would overestimate the volatility since coupons dampen a bond's price volatility. Duration is however a modified measure of maturity that reflects the reduced maturity due to the early payment of interest (coupons) prior to maturity. In particular, the duration of a coupon bond has the same price sensitivity as a zero coupon bond that has a maturity equal to the coupon bond's duration. Thus it follows (without calculus even) that $\% \Delta P = -Duration \times \Delta r / (1 + r)$ for a coupon bond. Duration may be used to predict price changes for small interest rate changes for coupon bonds. For convenience, practitioners sometimes calculate what is called '*modified duration*' (*MD*) which is $Duration / (1 + r)$ so that the only variable to be added to predict the price change is Δr . That is,

$$\frac{\Delta P}{P} = -MD(\Delta r), \quad \text{where } MD = \frac{D}{(1 + r)} \quad (\text{A-3})$$

There are several applications that you can use the duration to manage your portfolio:

- (a) Given the individual bond durations, the duration of a portfolio is a simple linear weighted average of the durations of the bonds in the portfolio. Using the portfolio's duration makes it very easy to predict the net value change of the portfolio if interest rates change.
- (b) We can use duration to measure the price sensitivity of different bonds to small changes in interest rates.
- (c) Duration measures the price volatility of a bond, but reinvestment rates also affect bond returns. For a given investment horizon, one can try to lock in the current promised yield to maturity by

choosing a bond with a duration equal to your investment horizon. This is a standard institutional bond investment strategy called ‘immunization.’ However, one can also try to beat the promised yield. If interest rates are projected to fall one could choose a bond with a duration greater than the investment horizon. If you are correct and rates fall, the gain in sale price of the bond will more than outweigh the lost reinvestment income caused by the lower reinvestment rate and your overall realized rate of return will be greater than the promised yield. Conversely, if you are projecting rising rates you can beat the promised yield by choosing a duration shorter than your investment horizon.

D. Large Interest Rate Changes and Duration

Duration is an accurate predictor of price changes only for very small interest rate changes. For day to day fluctuations duration works quite well but when interest rates move significantly, such as when the Fed makes an announcement of a rate change, the pricing errors can become significant.

Duration also predicts that the price changes of bonds are symmetric with respect to an interest rate increase and an interest rate decrease. An examination of Text Figure 3-7 indicates that this is not true. The bond’s price with respect to interest rates is convex to the origin. The duration is the first derivative or slope of that line (see Figure 3-7). Hence, the error in the bond price prediction is due to the curvature of the line and the degree of curvature is called the *convexity*. The greater the convexity is, the greater the error commits. This is due to the market behavior in that the lower the level of interest rates, the larger a bond’s price sensitivity to interest rate changes.

Construction of the yield curve can be found in the following website:

<http://www.treas.gov/offices/domestic-finance/debt-management/interest-rate/yieldmethod.html>

Daily Treasury Yield Curve Rates and historical data

http://www.treas.gov/offices/domestic-finance/debt-management/interest-rate/yield_historical_main.shtml