MEM633 Lectures 1&2

Chapter 1 Introduction

1-1 Properties of Systems

$$y(t)=H[u(t)]$$

A system H is said to be *linear* if and only if

$$H[c_1u_1+c_2u_2]=c_1H[u_1]+c_2H[u_2]$$

where c_1 and c_2 are *arbitrary* real or complex numbers

A system H is said to be *time-invariant* if and only if

$$H[u(t-\tau)]=y(t-\tau)$$

for *any* u(t) and *any* τ.

A system H is said to be *causal* if and only if its response to an input does not depend on future values of that input.

A system H is said to be *instantaneous* if its output is a function of the input *at the present time only*.

A system H is said to be *dynamic* if its output depends on *past and present values of the input*.

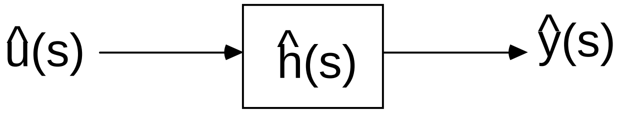
1-2 Representations of Systems

1. Differential Equations

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ . \ . \ . + a_0\, y(t)$$

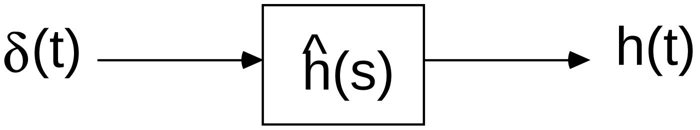
$$= b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \ . \ . \ . + b_0\, u(t$$

2. Transfer Functions



$$\hat{h}(s) = \hat{y}(s) \, / \, \hat{u}(s)$$

3. Impulse Response



$$L\ [\,h(t)\,] = \hat{h}(s)$$

$$y(t) = \int_{-\infty}^{t} h(t-\lambda)u(\lambda)d\lambda$$

4. State-space Representation

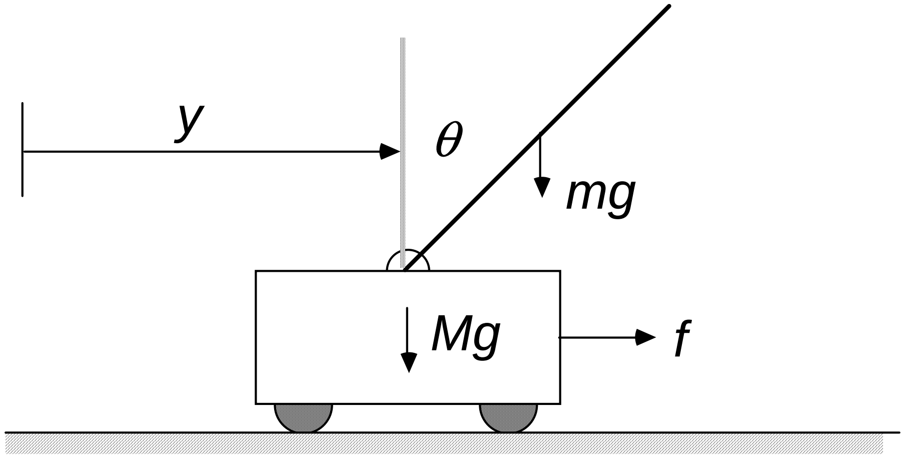
x(t): state vector

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t)$$

1-3 An Example

Inverted pendulum position system



Governing equations:

After linearization:

Choose state variables as

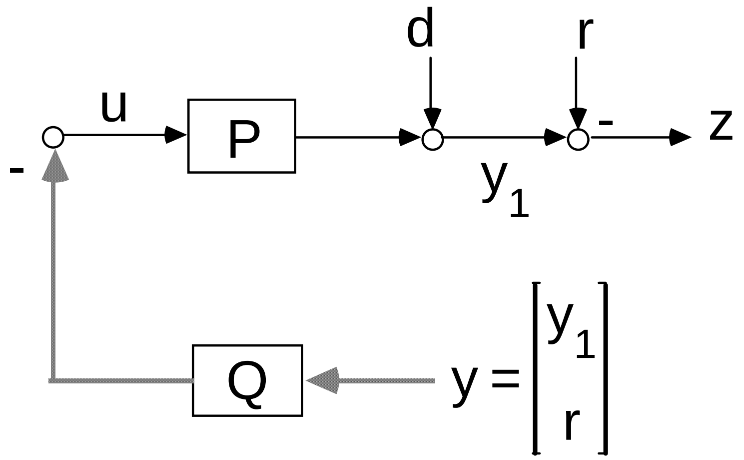
$x_1 = \theta\ ,\ x_2 = \dot{\theta}\ ,\ x_3 = y\ ,\ x_4 = \dot{y}$

Then

State equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\beta \\ 0 \\ \delta \end{bmatrix} f$$

1-4 Control Problems



P: plant, i.e., a system to be controlled.

y_1: system output vector

$$y_1 = P u + d.$$

d: disturbance input vector.

u: control input vector.

r: reference input vector, command.

z: controlled output, the output to be controlled

$$z = y_1 - r.$$

y: measured output, i.e., a vector which consists of all the variables can be measured.

Q: a controller (compensator) to be found.

$$u = - Qy.$$

Objectives:

To find a realizable controller Q such that the closed-loop system is internally stable and has some desired performance.

Make z as small as possible.

What does it meant by "a small z"?

What kind of disturbances and references we are dealing with?

Constraints on the control inputs?

Robust stability.

System remains stable under plant perturbations.

Robust performance.

z remains "small" under plant perturbations.

1-5 State Equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

Choose a new state vector $\bar{x}(t)$ as

$$\bar{x}(t) = T^{-1}x(t)$$

where T is a nonsingular matrix. Then

This transformation is referred as <u>similarity transformation</u>.

A system can have many state-space descriptions.

Diagonalization

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

To find a nonsingular matrix T such that $T^{-1}AT$ is a diagonal matrix.

Suppose we can find a nonsingular matrix T such that

$$T^{-1}AT = A_d = \text{diag}\,[\lambda_1, \lambda_2, \ldots., \lambda_n]$$

i.e.,

$$A\,T = T\,A_d$$

Let e_i, $i = 1,2,...,n$ be column vectors of T, i.e.,

$$T = [\,e_1 \;\; e_2 \;\; \ldots. \;\; e_n\,]$$

then

$$A\,e_i = \lambda_i\,e_i\,, \qquad i = 1,2,...,n$$

It is clear that λ_i must be an eigenvalue of A and e_i a corresponding eigenvector. Hence, a nonsingular T can be found if and only if A has n linearly independent eigenvectors.

If λ_1 , λ_2 , ..., λ_n are distinct, then their corresponding eigenvectors e_1 , e_2 , ..., e_n are linearly independent. But converse is not necessarily true.

Solution of the State Equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

After taking Laplace transform and some simple manipulations, we have

$$X(s) = (sI - A)^{-1}x(0-) + (sI - A)^{-1}BU(s)$$

The solution x(t) is simply the inverse Laplace transform of X(s), or

$$x(t) = e^{At}\,x(0-) + e^{At} * Bu(t)$$

State transition matrix

$$e^{At} = L^{-1}\left[(s\mathrm{I} - A)^{-1} \right]$$

$$e^{At} = \mathrm{I} + At + \frac{(At)^2}{2!} + \,.....\, + \frac{(At)^n}{n!} + \,....$$

By Cayley-Hamilton Theorem,

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t)\, A^k$$

where $\alpha_k(t)$, k=0,1,2,...,n-1, can be solved from

$$e^{\lambda_i t} = \sum_{k=0}^{n-1} \alpha_k(t)\, \lambda_i^k \qquad i = 1,2,....,n$$

Transfer functions

$$X(s) = (sI - A)^{-1}x(0-) + (sI - A)^{-1}BU(s), \qquad Y(s)=CX(s)$$

The transfer function matrix is

$$H(s) = C\,(sI - A)^{-1}B$$

and the impulse response matrix is

$$H(t) = C\,e^{At}\,B$$

In scalar case, the expression for H(s) can be written as

$$c\,(sI - A)^{-1}\,b = c\cdot\mathrm{Adj}\,(sI - A)\cdot b\,/\,\det\,(sI - A)$$
$$= b(s)/a(s)$$

b(s) and a(s) may have some common factors so that we can write

$$b(s)/a(s) = b_r(s)/a_r(s)$$

where

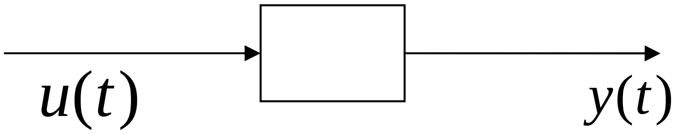
$$\{\, b_r(s), a_r(s) \,\} \text{ are relatively prime}$$

i.e., have no common factors (except possibly constants). The <u>poles</u> and <u>zeros</u> of H(s) are defined as the roots of the polynomials $a_r(s)$ and $b_r(s)$, respectively.

The definitions of poles and zeros of multivariable systems will be given later.

1-6 Stability

BIBO stability (External stability)



Def: A system is said to be BIBO stable (externally stable) if for each $M_1 < \infty$ there exists $M_2 < \infty$ such that

$$| u(t) | \le M_1 \quad \text{implies} \quad | y(t) | \le M_2$$

Theorem: A linear time-invariant system with impulse response h(t) is BIBO stable if and only if

$$\int_0^\infty |h(t)| dt = M < \infty$$

Theorem: A linear time-invariant system with transfer function H(s) = b(s)/a(s) is BIBO stable if and only if all the poles of H(s) lie in the strictly left half of s-plane, i.e., the real parts of all the poles are negative.

Remark: Suppose b(s), a(s) are coprime, then

poles of H(s) = zeros of a(s).

Internal stability

Def: The linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is internally stable if the solution x(t) of

$$\dot{x}(t) = A\,x(t) \quad \text{with initial state } x(0)$$

tends toward zero as $t \rightarrow \infty$ for arbitrary x(0).

Theorem:

A linear time-invariant system is internally stable if and only if all the eigenvalues of A matrix have negative real parts.

Unstable pole-zero cancellation

Consider a system with transfer function

$$P(s) = \frac{1}{s - 1}$$

This system is unstable. To stabilize it, let's try the compensation technique shown in Fig.1.6-1 with the compensator

$$C(s) = \frac{s - 1}{s + 1}$$

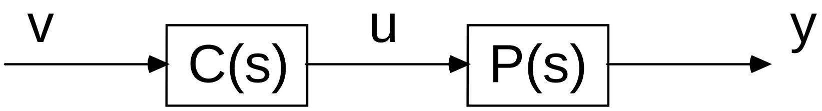


Fig. 1.6-1

Then we have the overall transfer function

$$P(s)C(s) = \frac{1}{s - 1}\,\frac{s - 1}{s + 1} \overset{?}{=} \frac{1}{s + 1}$$

It looks nice, but unfortunately this technique will not work!!

In practice, it is difficult to ensure exact cancellation because of variations in component values, etc.

Even with perfect cancellation, this technique still does not work. To see why, let's first set up an analog-computer simulation as shown in Fig.1.6-2.

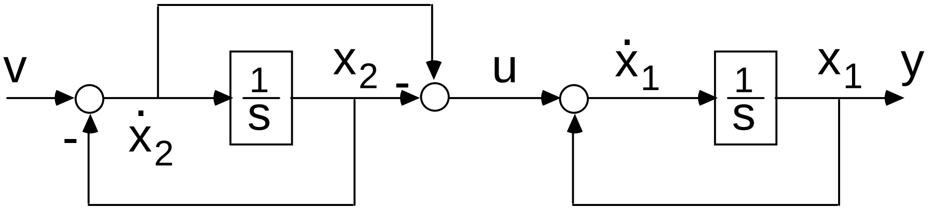


Fig. 1.6-2

Then we have the state equations

$$\dot{x}_1(t) = x_1(t) - 2\,x_2(t) + v(t), \qquad x_1(0) = x_{10}$$

$$\dot{x}_2(t) = -\,x_2(t) + v(t), \qquad x_2(0) = x_{20}$$

and the output equation

$$y(t) = x_1(t)$$

By solving these equations, we have

$$y(t) = x_1(t) = x_{10}\, e^{t} + x_{20}\, (e^{-t} - e^{t}) + e^{-t} * v(t)$$

where $*$ denotes convolution. We can see that the output y(t) will grow without bound unless the initial conditions can always be kept zero.

Now, let's try the following

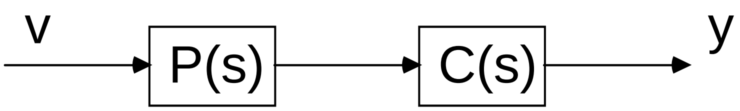


Fig. 1.6-3

The analog-computer simulation of the system is shown in Fig.1.6-4.

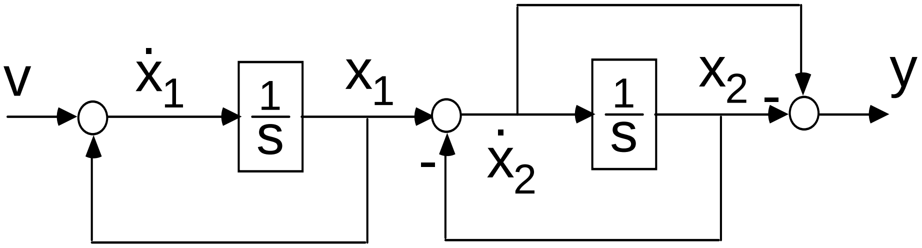


Fig. 1.6-4

Then we have the dynamic equations

$$\dot{x}_1(t) = x_1(t) + v(t), \quad x_1(0) = x_{10}$$

$$\dot{x}_2(t) = x_1(t) - x_2(t), \quad x_2(0) = x_{20}$$

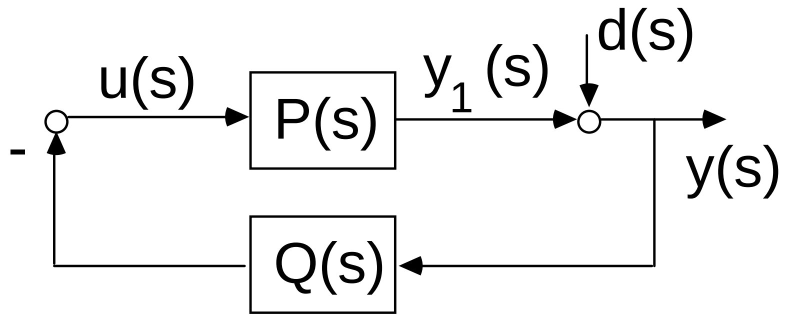
$$y(t) = x_1(t) - 2x_2(t)$$

The solution is

$$y(t) = x_{10}\, e^{-t} - x_{20}\, e^{-t} + e^{-t} * v(t)$$

y(t) looks O.K., but the system is still internally unstable.

Feedback Connection

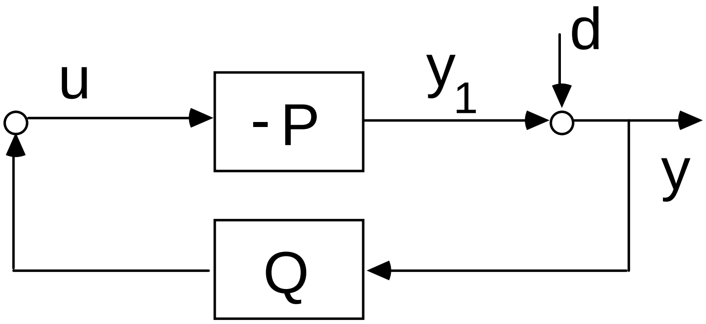


The transfer matrix from d(s) to y(s) is

$$[\, I + P(s)Q(s) \,]^{-1}$$

The feedback system is BIBO stable iff all the poles of $[\, I + P(s)Q(s) \,]^{-1}$ are in LHP, i.e., all the zeros of det $[\, I + P(s)Q(s) \,]$ are in the LHP.

State-space representation



- P :
$$\dot{x}_P = A_P x_P + B_P u$$
$$y_1 = C_P x_P$$

Q :
$$\dot{x}_Q = A_Q x_Q + B_Q y$$
$$u = C_Q x_Q$$

Define

$$x = \begin{bmatrix} x_P \\ x_Q \end{bmatrix}$$

then

The closed-loop system is internally stable iff all the zeros of

$$\det \begin{bmatrix} sI - A_P & -B_P C_Q \\ -B_Q C_P & sI - A_Q \end{bmatrix}$$

are in LHP.

Theorem:

Let $\phi_P(s)$ and $\phi_Q(s)$ be characteristic polynomals of systems P and Q respectively, i.e.,

$$\phi_P(s) = \det [\, sI - A_P],\ \ \phi_Q(s) = \det [\, sI - A_Q].$$

Then the closed-loop system is internally stable iff all the zeros of

$$\phi_P(s)\, \phi_Q(s) \det [\, I + P(s)Q(s)\,]$$

are in the LHP (i.e., with strictly negative real parts).

Lemma: X, W are Invertible, then

$$\det \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = |X|\,|W|\,|\,I - ZX^{-1}YW^{-1}|$$

Lemma:

Let M : mxn matrix

N : nxm matrix

I_m : mxm identity matrix

I_n : nxn identity matrix

Then

$$| I_m - MN | = | I_n - NM |$$

Proof of the Theorem:

Remark:

Internal stability $\Rightarrow$ BIBO stability.

$\not\Leftarrow$